

On Kirchhoff type equations with critical Sobolev exponent and Naimen's open problems

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Abstract

We study the following Brezis-Nirenberg problem of Kirchhoff type

$$\begin{cases} -(a + b \int_{\Omega} |\nabla u|^2 dx) \Delta u = \lambda |u|^{q-2} u + \delta |u|^2 u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^4$ is a bounded domain with the smooth boundary $\partial\Omega$, $2 \leq q < 4$ and a, b, λ, δ are positive parameters. We obtain some new existence and nonexistence results, depending on the values of the above parameters, which improves some known results. The asymptotical behaviors of the solutions are also considered in this paper.

Keywords: Kirchhoff type equation; nonlocal problem; Brezis-Nirenberg problem; perturbation method.

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1 Introduction

In this paper, we consider the following Kirchhoff type problem involving critical nonlinearity

$$\begin{cases} -(a + b \int_{\Omega} |\nabla u|^2 dx) \Delta u = \lambda |u|^{q-2} u + \delta |u|^2 u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^4$ is a bounded domain with the smooth boundary $\partial\Omega$, $2 \leq q < 4$, a, b, λ, δ are positive parameters.

Eq. (1.1) is related to the stationary of the Kirchhoff type quasilinear hyperbolic equation

$$u_{tt} - \left(a + b \int_{\Omega} |\nabla u|^2 dy \right) \Delta u = f(x, u), \quad \text{in } \Omega, \quad (1.2)$$

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that was first proposed by Kirchhoff in [21] describing the transversal oscillations of a stretched string. For more details on the physical and mathematical background of Eq. (1.2), we refer the readers to the papers [2, 21, 22, 24] and the references therein. Besides the physical motivation, such problems are often referred to as being nonlocal because of the presence of the term $\int_{\Omega} |\nabla u|^2 dx$ which implies that the equation in (1.1) is no longer a pointwise identity. This phenomenon leads to some mathematical difficulties, which makes the study of such a class of problems particularly interesting. Equations like (1.1) has been studied extensively by using variational methods recently, see [2, 3, 4, 15, 16, 17, 18, 23, 24, 25, 26, 27, 33, 34, 35, 36] and the references therein.

In the study of Eq. (1.1), our main concern is focused on the nonlinearities with critical growth starting from the pioneering paper [6] by Brezis and Nirenberg, in which (1.1) is studied in the case of $a = \delta = 1$ and $b = 0$. Many mathematicians have paid great attention to this type of critical problems for their stimulating and challenging difficulties coming from the lack of compactness of the Sobolev imbedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$, see e.g. [8, 9, 10, 11, 14, 31, 32, 37] and the references therein for the existence and multiplicity results. Recently the Brezis-Nirenberg problem of Kirchhoff type is investigated in [3, 16, 25, 26] and the references therein.

In order to describe better our results, we distinguish two cases of $q = 2$ and $2 < q < 4$ according to the range of q .

1.1 The case of $q = 2$ and $\lambda \geq a\lambda_1$, i.e. the indefinite case of Kirchhoff type Brezis-Nirenberg problem

In this case, Eq. (1.1) becomes the following Brezis-Nirenberg problem involving Kirchhoff type nonlocal term

$$\begin{cases} -(a + b \int_{\Omega} |\nabla u|^2 dx) \Delta u = \lambda u + \delta |u|^2 u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.3)$$

which has been proved to have a solution for $\lambda \in (0, a\lambda_1)$ if and only if $0 \leq b\mathcal{S}^2 < \delta$ by Naimen in [25] recently, where λ_1 is the first eigenvalue of $-\Delta$ on Ω and \mathcal{S} is the best Sobolev constant defined in (2.2). Then some natural questions are arisen: what will happen if $\lambda \geq a\lambda_1$? What about the asymptotical behaviors of the solutions depending on the parameters? The first aim of this paper is to try to answer these questions. Note that, compared with the case of $\lambda < a\lambda_1$, the case of $\lambda \geq a\lambda_1$ is more sophisticated because the operator $-a\Delta - \lambda$ is indefinite. In order to deal with this case, besides the typical difficulty caused by the lack of compactness of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$, we will face the following three difficulties: the first is to prove the boundedness of the Palais-Smale sequence (the (PS) sequence in short); the second is the interaction between the Kirchhoff type perturbation $\|u\|_{H_0^1(\Omega)}^4$ and the critical nonlinearity $\int_{\Omega} |u|^4 dx$, in particular, they have the same exponent; the third is that the weak limit of the bounded (PS) sequence cannot be seen as the weak solution of Eq. (1.3) directly.

According to the relationship between $b\mathcal{S}^2$ and δ , we will consider the following two cases: $0 < \delta \leq b\mathcal{S}^2$ and $0 < b\mathcal{S}^2 < \delta$. For the first case, in order to study the asymptotical behaviors of

the solutions, we give some existence results of solutions for the following eigenvalue problem

$$\begin{cases} -(a + b \int_{\Omega} |\nabla u|^2 dx) \Delta u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \quad (1.4)$$

Theorem 1.1 *Suppose that $a > 0$, $b > 0$ and $\lambda > 0$.*

- (I) *If $\lambda \in (0, a\lambda_1]$ then Eq. (1.4) has no nontrivial solution.*
- (II) *If $\lambda \in (a\lambda_1, a\lambda_2]$ then Eq. (1.4) has a unique positive ground state solution $u = \sqrt{\frac{\lambda - a\lambda_1}{b\lambda_1^2}} \varphi_{11}$.*
- (III) *If $\lambda \in (a\lambda_k, a\lambda_{k+1}]$ with $k \geq 2$ then Eq. (1.4) has a unique positive ground state solution $u = \sqrt{\frac{\lambda - a\lambda_1}{b\lambda_1^2}} \varphi_{11}$; moreover, Eq. (1.4) has sign-changing solutions $\bar{u}_j = \sqrt{\frac{\lambda - a\lambda_j}{b\lambda_j^2}} \psi_j$ for all $\psi_j \in M(\lambda_j)$ satisfying*

$$\psi_j = \sum_{i=1}^{i_j} c_{ij} \varphi_{ij} \text{ with } \sum_{i=1}^{i_j} c_{ij}^2 = 1, \quad j = 2, 3, \dots, k,$$

where $M(\lambda_j)$, φ_{ij} and i_j will be given in Section 2.

Remark 1.1 As far as we know, there is no result on the eigenvalue problem (1.4) in literatures. Comparing to the well-known results in the case $b = 0$, we see that Eq. (1.4) also has infinitely many sign-changing solutions provided $\lambda \in (a\lambda_k, a\lambda_{k+1}]$ with $k \geq 2$. Eq. (1.4) can be seen as the limited problem of (1.3) as $\delta \rightarrow 0^+$, which will help us to study the asymptotical behaviors of the solutions of (1.3).

Now we give some existence and nonexistence results of solutions for Eq. (1.3) in the case of $0 < \delta \leq bS^2$.

Theorem 1.2 *Suppose that $a > 0$, $b > 0$, $\lambda > 0$ and $\delta > 0$ with $0 < \delta \leq bS^2$.*

- (1) *If $\lambda \in (0, a\lambda_1]$ then Eq. (1.3) has no nontrivial solution.*
- (2) *If $\lambda > a\lambda_1$ and $\delta < bS^2$ then Eq. (1.3) has one positive ground state solution.*
- (3) *Assume that $\lambda \in (a\lambda_k, a\lambda_{k+1}]$ with $k \geq 1$ and $\{\delta_n\}$ is a sequence of positive numbers satisfying $\delta_n \rightarrow 0$ as $n \rightarrow \infty$. Let u_n be the positive ground state solution corresponding to δ_n , then there exists $u_0 \in H_0^1(\Omega) \setminus \{0\}$ such that $u_n \rightarrow u_0$ in $H_0^1(\Omega)$ as $n \rightarrow \infty$, up to a subsequence. Moreover,*

$$u_0 = \sqrt{\frac{\lambda - a\lambda_1}{b\lambda_1^2}} \varphi_{11}$$

is a positive ground state solution of the problem of (1.4).

Remark 1.2 The conclusions of (3) of Theorem 1.2 gives a description of asymptotical behavior of the positive ground state solution of (1.3) as $\delta \rightarrow 0^+$, however, an interesting problem is that, besides the positive ground state solution, is there any solution of (1.3) converging to a sign-changing solution of (1.4) as $\delta \rightarrow 0^+$ if $\delta < bS^2$?

For the case of $0 < b\mathcal{S}^2 < \delta$, we have the following results.

Theorem 1.3 *Suppose that $a > 0$, $b > 0$, $\lambda > 0$ and $\delta > 0$ with $\delta > b\mathcal{S}^2$.*

(i) *If*

$$a\lambda_1 \leq \lambda \leq \frac{a\lambda_1\delta}{\delta - b\mathcal{S}^2} \quad (1.5)$$

then Eq. (1.3) has at least one pair of sign-changing solutions under one of the following conditions:

(1) $\delta > 0$, $b > 0$ *is sufficiently small;*

(2) $b > 0$, $\delta > 0$ *is sufficiently large.*

(ii) *For $a, \delta > 0$ and let $\{b_n\}$ and $\{\bar{\lambda}_n\}$ be two sequences of positive numbers satisfying $b_n \rightarrow 0$ and $\bar{\lambda}_n \rightarrow a\lambda_1$ as $n \rightarrow \infty$. Assume that u_n is the solution corresponding to b_n and $\bar{\lambda}_n$ obtained above, then there exists $u_0 \in H_0^1(\Omega) \setminus \{0\}$ such that $u_n \rightarrow u_0$ in $H_0^1(\Omega)$ as $n \rightarrow \infty$, up to a subsequence, and u_0 is a sign-changing solution of the following equation:*

$$\begin{cases} -a\Delta u = a\lambda_1 u + \delta|u|^2 u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \quad (1.6)$$

Remark 1.3 (i) From our results, we see that when $\lambda \geq a\lambda_1$, the cases of $b = 0$ and $b > 0$ are quite different.

(ii) Theorems 1.2 and 1.3 complement and generalize the results of [25, Theorem 1.1] and [9, Theorem 0.1]. The assumption (1.5) can be removed if $\lambda = a\lambda_1$. Moreover, the asymptotical behaviors of the solutions are respectively given in Theorem 1.2 as $\delta \rightarrow 0$ and in Theorem 1.3 as $b \rightarrow 0$.

(iii) According to [11, 14], Eq. (1.6) has multiple nontrivial solutions if we take the place of λ_1 by λ , moreover the existence of ground state solutions of Eq. (1.6) also obtained in [30, 11], however, both of the existence of ground state solutions and the multiplicity of nontrivial solutions of (1.3) are still unknown for $\lambda \geq a\lambda_1$ and $\delta > b\mathcal{S}^2$.

1.2 The case of $2 < q < 4$ and Naimen's open question.

The case of $2 < q < 4$ is more thorny and tough because the boundedness of the (PS) sequences is hard to prove. To overcome this difficulty, by using the well-known monotonicity trick due to Struwe (see also [19, 29]), Naimen [25] obtained the following theorem:

Theorem A. ([25, Theorem 1.6]) *Suppose that $2 < q < 4$. Let $b, \delta > 0$ satisfy $b\mathcal{S}^2 < \delta < 2b\mathcal{S}^2$ and let $\Omega \subset \mathbb{R}^4$ be strictly star shaped. Furthermore, assume that one of the following conditions (C1), (C2) and (C3) holds:*

(C1) $a > 0$, $\lambda > 0$ *is small enough,*

(C2) $\lambda > 0$, $a > 0$ *is large enough,*

(C3) $a > 0$, $\lambda > 0$ and $\delta/b > \mathcal{S}^2$ *is sufficiently close to \mathcal{S}^2 ,*

then Eq. (1.1) has a nontrivial solution.

Naimen's open question. On [25, Page 1171], Naimen asked whether conditions that $\mu < 2b\mathcal{S}^2$, $\Omega \subset \mathbb{R}^4$ is strictly star-shaped and (C1)-(C3) in Theorem A are necessary to ensure the existence of the solutions of Eq. (1.1).

Another purpose of the present paper is to try to give an answer to Naimen's open question. In order to do this, inspired by [7, 12, 13, 20], we will construct a bounded (PS) sequence and show that the critical level of the functional is below the compactness threshold $\frac{(a\mathcal{S})^2}{4(\delta - b\mathcal{S}^2)}$ by applying a perturbation method, then we obtain the following result which gives a partial answer to Naimen's open question.

Theorem 1.4 *Suppose that $2 < q < 4$, $a > 0$, $b > 0$, $\lambda > 0$ and $\delta > 0$ with $\delta > b\mathcal{S}^2$. Then there exists $b_1^* > 0$ such that for each $b \in (0, b_1^*)$, Eq. (1.1) has a solution if one of the above conditions (C1)-(C3) is satisfied.*

It is worth observing that Theorem 1.4 is complementary to the corresponding result of [16, Theorem 1.1], where $\lambda > 0$ is assumed to be large enough.

Furthermore, according to Theorem 1.4, a natural questions about Eq.(1.1) can be arisen: what will happen if $0 < \delta \leq b\mathcal{S}^2$? For this question, we have the following nonexistence and multiplicity results.

Theorem 1.5 *Suppose that $2 < q < 4$, $a > 0$, $b > 0$, $\lambda > 0$ and $\delta > 0$ with $\delta \leq b\mathcal{S}^2$.*

(i) *If $\delta < b\mathcal{S}^2$ and*

$$a \geq \frac{(4-q)|\Omega|^{\frac{1}{2}}}{\mathcal{S}} \left(\frac{\lambda}{2}\right)^{\frac{2}{4-q}} \left(\frac{q-2}{b\mathcal{S}^2 - \delta}\right)^{\frac{q-2}{4-q}} \quad (1.7)$$

then Eq. (1.1) has no nontrivial solution;

(ii) *there exists $b_2^* > 0$ dependent of a and λ such that for each $b \in (0, b_2^*)$, Eq. (1.1) has at least two nontrivial solutions provided $\delta < b\mathcal{S}^2$ and has at least one nontrivial solution provided $\delta = b\mathcal{S}^2$.*

This paper is organized as follows. In Section 2 we give some preliminaries. In Section 3 we give a proof of Theorem 1.1. Section 4 is devoted to prove Theorems 1.2 and 1.3 by using Ekeland variational principle and applying an idea of [9], in order to overcome the interaction between the nonlocal term and the critical nonlinearity, we give some new estimates (see Lemma 4.4). In Section 5 we prove Theorems 1.4 and 1.5 by using a perturbation method introduced in [7, 20].

2 Preliminaries

Throughout this paper, we will use the following common notations.

Notations:

- $L^p(\Omega)$ is the usual Lebesgue space with the norm $\|u\|_p = \left(\int_{\Omega} |u|^p dx\right)^{1/p}$.
- $H_0^1(\Omega)$ and $\mathcal{D}^{1,2}(\mathbb{R}^4)$ are the usual Sobolev space with the associated norms $\|u\| = \left(\int_{\Omega} |\nabla u|^2 dx\right)^{1/2}$ and $\|u\|_{\mathcal{D}} = \left(\int_{\mathbb{R}^4} |\nabla u|^2 dx\right)^{1/2}$ respectively.

- Denote by $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j < \cdots, j \in \mathbb{N}$ the eigenvalues of $(-\Delta, H_0^1(\Omega))$ and by $M(\lambda_j)$ the corresponding eigenspace of λ_j in $H_0^1(\Omega)$.
- $\{\varphi_{ij}\}$ is the orthogonal eigenfunctions corresponding to λ_j spanning $M(\lambda_j)$ and $\int_{\Omega} |\varphi_{ij}|^2 dx = 1, i = 1, 2, \dots, i_j$, where $i_j = \dim M(\lambda_j)$.
- $\Omega \subset \mathbb{R}^N$ is a bounded domain and denote $|\Omega|$ by the Lebesgue measure of Ω .
- C, C_1, C_2, \dots denote the positive (possibly different) constants.

We first note that the corresponding functional of (1.1) $I : H_0^1(\Omega) \rightarrow \mathbb{R}$ is given by

$$I(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\lambda}{q}\|u\|_q^q - \frac{\delta}{4}\|u\|_4^4, \quad \forall u \in H_0^1(\Omega). \quad (2.1)$$

Clearly, $I \in C^1(H_0^1(\Omega), \mathbb{R})$ and

$$I'(u)v = (a + b\|u\|^2) \int_{\Omega} \nabla u \nabla v dx - \lambda \int_{\Omega} |u|^{q-2} uv dx - \delta \int_{\Omega} |u|^2 uv dx, \quad \forall u, v \in H_0^1(\Omega).$$

Recalling that the best Sobolev constant \mathcal{S} of $\mathcal{D}^{1,2}(\mathbb{R}^4) \hookrightarrow L^4(\mathbb{R}^4)$ defined as

$$\mathcal{S} = \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^4) \setminus \{0\}} \frac{\int_{\mathbb{R}^4} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^4} |u|^4 dx \right)^{1/2}} \quad (2.2)$$

can be achieved by the Talenti function

$$U_{\varepsilon}(x) := \frac{(8\varepsilon)^{1/2}}{\varepsilon + |x - x_0|^2}, \quad (2.3)$$

which is the positive solution of the problem in whole space

$$-\Delta U = U^3, \quad \text{in } \mathbb{R}^4, \quad U(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty$$

for some $\varepsilon > 0$ and $x_0 \in \mathbb{R}^4$. As a matter of fact, when $b\mathcal{S}^2 < \delta$, the Talenti function multiplied by an appropriate constant,

$$W_{\varepsilon} := \left(\frac{a}{\delta - b\mathcal{S}^2} \right)^{1/2} U_{\varepsilon}$$

is nothing but a solution of the Kirchhoff type equation in whole space

$$-(a + b \int_{\mathbb{R}^4} |\nabla W|^2) \Delta W = \delta W^3, \quad \text{in } \mathbb{R}^4, \quad W(x) \rightarrow 0, \text{ as } |x| \rightarrow \infty. \quad (2.4)$$

Moreover, we can easily check that the functional energy of W_{ε} satisfies

$$\frac{a}{2}\|W_{\varepsilon}\|_{\mathcal{D}}^2 + \frac{b}{4}\|W_{\varepsilon}\|_{\mathcal{D}}^4 - \frac{\delta}{4} \int_{\mathbb{R}^4} |W_{\varepsilon}|^4 dx = \frac{(a\mathcal{S}^2)}{4(\delta - b\mathcal{S}^2)}.$$

However, it is easy to see that (2.4) has no nontrivial solutions if $0 < \delta \leq b\mathcal{S}^2$. Indeed, if not, we assume that $W \in \mathcal{D}^{1,2}(\mathbb{R}^4) \setminus \{0\}$ is a solution of (2.4), then we obtain

$$0 = a\|W\|_{\mathcal{D}}^2 + b\|W\|_{\mathcal{D}}^4 - \delta \int_{\mathbb{R}^4} |W|^4 dx \geq a\|W\|_{\mathcal{D}}^2 + (b - \frac{\delta}{\mathcal{S}^2})\|W\|_{\mathcal{D}}^4 \geq a\|W\|_{\mathcal{D}}^2 > 0, \quad (2.5)$$

this is a contradiction.

The following result can be obtained using the mathematical induction.

Lemma 2.1 *Let $x_k, y_k \in \mathbb{R}^+$ ($k = 1, 2, \dots, n$) be positive constants, then it holds*

$$\frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n y_i} \leq \max \left\{ \frac{x_k}{y_k}, k = 1, 2, \dots, n \right\}.$$

Next, we list a global compactness result which gives the description of $(PS)_c$ sequence (cf. [25]).

Proposition 2.1 *Let $c \in \mathbb{R}$ and $\{u_n\} \subset H_0^1(\Omega) \subset \mathcal{D}^{1,2}(\mathbb{R}^4)$ be a bounded $(PS)_c$ for I , that is, $I(u_n) \rightarrow c$, $I'(u_n) \rightarrow 0$ in $H^{-1}(\Omega)$ and $\{u_n\}$ is bounded. Then $\{u_n\}$ has a subsequence which strongly converges in $H_0^1(\Omega)$, or otherwise, there exist a function $u_0 \in H_0^1(\Omega)$ which is a weak convergence of $\{u_n\}$, a number $k \in \mathbb{N}$ and further, for every $i \in \{1, 2, \dots, k\}$, a sequence of value $\{R_n^i\}_{n \in \mathbb{N}} \subset \mathbb{R}^+$, points $\{x_i\}_{n \in \mathbb{N}} \subset \overline{\Omega}$ and a function $v_i \in \mathcal{D}^{1,2}(\mathbb{R}^4)$ satisfying*

$$- \left\{ a + b \left(\|u_0\|^2 + \sum_{j=1}^k \|v_j\|_{\mathcal{D}}^2 \right) \right\} \Delta u_0 = \lambda u_0^q + \delta u_0^3, \quad \text{in } \Omega, \quad (2.6)$$

$$- \left\{ a + b \left(\|u_0\|^2 + \sum_{j=1}^k \|v_j\|_{\mathcal{D}}^2 \right) \right\} \Delta v_i = \delta v_i^3, \quad \text{in } \mathbb{R}^4, \quad (2.7)$$

such that up to subsequences, there hold $R_n^i \text{dist}(x_n^i, \partial\Omega) \rightarrow \infty$,

$$\left\| u_n - u_0 - \sum_{i=1}^k R_n^i v_i(R_n^i(\cdot - x_n^i)) \right\|_{\mathcal{D}} = o(1),$$

$$\|u_n\|^2 = \|u_0\|^2 + \sum_{i=1}^k \|v_i\|_{\mathcal{D}}^2 + o(1)$$

and

$$I(u_n) = \tilde{I}(u_0) + \sum_{i=1}^k \tilde{I}^\infty(v_i) + o(1)$$

where $o(1) \rightarrow 0$ as $n \rightarrow \infty$ and we define

$$\tilde{I}(u_0) := \left\{ \frac{a}{2} + \frac{b}{4} \left(\|u_0\|^2 + \sum_{j=1}^k \|v_j\|_{\mathcal{D}}^2 \right) \right\} \|u_0\|^2 - \frac{\lambda}{q} \int_{\Omega} |u_0|^q dx - \frac{\delta}{4} \int_{\Omega} |u_0|^4 dx, \quad (2.8)$$

$$\tilde{I}^\infty(u_0) := \left\{ \frac{a}{2} + \frac{b}{4} \left(\|u_0\|^2 + \sum_{j=1}^k \|v_j\|_{\mathcal{D}}^2 \right) \right\} \|v_i\|_{\mathcal{D}}^2 - \frac{\delta}{4} \int_{\Omega} |v_i|^4 dx. \quad (2.9)$$

Finally, we recall a critical point theorem (cf. [5, Theorem 2.4]) which is a variant of some results contained in [1].

Theorem 2.1 *Let H be a real Hilbert space and $I \in C^1(H, \mathbb{R})$ be a functional satisfying the following assumptions:*

- (a₁) $I(u) = I(-u)$, $I(0) = 0$ for any $u \in H$;
- (a₂) there exists $\beta > 0$ such that I satisfies (PS) condition in $(0, \beta)$;
- (a₃) there exist two closed subspace $V, W \subset H$ and positive constants ρ, η such that

- (a₃₁) $I(u) < \beta$ for any $u \in W$;
- (a₃₂) $I(u) \geq \eta$ for any $u \in V$ with $\|u\| = \rho$;
- (a₃₃) $\text{codim}V < \infty$.

Then there exist at least m pair of critical points with the corresponding critical values in $[\eta, \beta)$ and

$$m = \dim(V \cap W) - \text{codim}(V \oplus W).$$

3 Proof of Theorem 1.1

Let the corresponding functional $J : H_0^1(\Omega) \rightarrow \mathbb{R}$ of (1.4) be defined by

$$J(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\lambda}{2}\|u\|_2^2, \quad u \in H_0^1(\Omega). \quad (3.1)$$

Clearly, the $J \in C^1(H_0^1(\Omega), \mathbb{R})$ and

$$J'(u)v = (a + b\|u\|^2) \int_{\Omega} \nabla u \nabla v dx - \lambda \int_{\Omega} uv dx, \quad u, v \in H_0^1(\Omega).$$

Therefore critical points of J are the weak solutions of (1.4).

Proof of of Theorem 1.1.

(I): Suppose on the contrary that Eq. (1.4) has a nontrivial solution $u \in H_0^1(\Omega) \setminus \{0\}$ for $\lambda \in (0, a\lambda_1]$, then we have

$$0 = a\|u\|^2 + b\|u\|^4 - \lambda\|u\|_2^2 \geq (a - \frac{\lambda}{\lambda_1})\|u\|^2 + b\|u\|^4 > 0,$$

this is impossible.

Before giving the proofs of (II) and (III), we shall prove the following claim:

Claim: If $\lambda \in (a\lambda_k, a\lambda_{k+1}]$ with $k \geq 1$ then u is a nontrivial solution of (1.4) if and only if u has the form of

$$\bar{u}_j := \sqrt{\frac{\lambda - a\lambda_j}{b\lambda_j^2}} \psi_j, \quad (3.2)$$

where $\psi_j \in M(\lambda_j)$ satisfying

$$\psi_j = \sum_{i=1}^{i_j} c_{ij} \varphi_{ij} \text{ and } \sum_{i=1}^{i_j} c_{ij}^2 = 1, \quad j = 1, 2, \dots, k.$$

Indeed, u is a nontrivial solution of (1.4) if and only if u is a solution of the eigenvalue problem

$$\begin{cases} -\Delta u = \frac{\lambda}{a + b\|u\|^2} u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Thus we must get

$$\frac{\lambda}{a + b\|u\|^2} = \lambda_j \text{ and } u = t_j \psi_j,$$

for some $t_j > 0$, $\psi_j \in M(\lambda_j)$ with $\|\psi_j\|_2^2 = 1$, i.e., $\lambda = a\lambda_j + b\lambda_j t_j^2 \|\psi_j\|^2 \leq a\lambda_{k+1}$, which implies that

$$t_j = \sqrt{\frac{\lambda - a\lambda_j}{b\lambda_j^2}}, \quad j = 1, 2, \dots, k,$$

so that

$$u = \sqrt{\frac{\lambda - a\lambda_j}{b\lambda_j^2}} \psi_j, \quad \psi_j \in M(\lambda_j) \text{ with } \psi_j = \sum_{i=1}^{i_j} c_{ij} \varphi_{ij} \text{ and } \sum_{i=1}^{i_j} c_{ij}^2 = 1, \quad j = 1, 2, \dots, k.$$

On the other hand, if u has the form of (3.2), then it is easy to see that u is a solution of (1.4). Therefore we complete the proof of this claim.

(II): For the case of $\lambda \in (a\lambda_1, a\lambda_2]$, since $\psi_1 = \varphi_{11}$, it follows from the above claim that (1.4) has the unique positive solution

$$\bar{u}_1 = \sqrt{\frac{\lambda - a\lambda_1}{b\lambda_1^2}} \varphi_{11}.$$

(III): For the case $\lambda \in (a\lambda_k, a\lambda_{k+1}]$, by the above claim, we see that (1.4) has the solution of the form (3.2). We will complete the proof of Theorem 1.1 by going through the following two steps.

Step 1. $J(\bar{u}_1) = \min\{J(\bar{u}_j) : j = 1, 2, \dots, k\}$.

Indeed, for $j \in \{1, 2, \dots, k\}$, we have

$$J(\bar{u}_j) = \frac{a(\lambda - a\lambda_j)}{2b\lambda_j^2} \|\psi_j\|^2 + \frac{b}{4} \left(\frac{\lambda - a\lambda_j}{b\lambda_j^2} \right)^2 \|\psi_j\|^4 - \frac{\lambda(\lambda - a\lambda_j)}{2b\lambda_j^2} \|\psi_j\|_2^2 = -\frac{(\lambda - a\lambda_j)^2}{4b\lambda_j^2}.$$

Set $g(t) = -(\lambda - at)^2/(4bt^2)$, $t \in (0, \lambda/a)$, by a direct calculation, we see that $g'(t) > 0$ for $t \in (0, \lambda/a)$, therefore, $g(\lambda_1) < g(\lambda_2) < \dots < g(\lambda_k)$ because $\lambda_1 < \lambda_2 < \dots < \lambda_k$, so that $J(\bar{u}_1) = \min\{J(\bar{u}_j) : j = 1, 2, \dots, k\}$.

Step 2. \bar{u}_j is sign-changing, $j = 2, 3, \dots, k$.

In fact, arguing by contradiction, without loss of generality, we may assume that $\bar{u}_j \geq 0$ for some $2 \leq j \leq k$. Recalling that $\varphi_{11} > 0$ in Ω and $\bar{u}_j \in M(\lambda_j)$ with

$$\bar{u}_j = \sqrt{\frac{\lambda - a\lambda_j}{b\lambda_j^2}} \psi_j = \sqrt{\frac{\lambda - a\lambda_j}{b\lambda_j^2}} \sum_{i=1}^{i_j} c_{ij} \varphi_{ij} \text{ and } \sum_{i=1}^{i_j} c_{ij}^2 = 1,$$

then we obtain

$$0 \leq \int_{\Omega} \bar{u}_j \varphi_{11} dx = \sqrt{\frac{\lambda - a\lambda_j}{b\lambda_j^2}} \sum_{i=1}^{i_j} c_{ij} \int_{\Omega} \varphi_{ij} \varphi_{11} dx = 0,$$

which means that $\bar{u}_j \equiv 0$, this is a contradiction since $\|\bar{u}_j\|_2^2 = (\lambda - a\lambda_j)/(b\lambda_j^2) > 0$. ■

4 Proof of Theorems 1.2 and 1.3

In this section, we are interested in the existence of solutions to (1.3) for $\lambda \geq a\lambda_1$. We note that the corresponding functional I given by (2.1) with $q = 2$ may be indefinite. We first establish a lemma which gives the description of $(PS)_c$ sequence of I for the above cases.

Lemma 4.1 Suppose that $a, \lambda, \delta > 0, b \geq 0$ satisfy

$$a\lambda_1 \leq \lambda \leq \frac{a\lambda_1\delta}{\delta - b\mathcal{S}^2}, \quad \frac{b|\Omega|\lambda^2}{a^2} < \delta.$$

If $\{u_n\} \subset H_0^1(\Omega)$ is a $(PS)_c$ sequence of I with $c \in (0, \frac{(a\mathcal{S})^2}{4(\delta - b\mathcal{S}^2)})$, then $\{u_n\}$ strongly converges in $H_0^1(\Omega)$ up to a subsequence.

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence for I with $c \in (0, \frac{(a\mathcal{S})^2}{4(\delta - b\mathcal{S}^2)})$, then

$$I(u_n) \rightarrow c \text{ and } I'(u_n) \rightarrow 0 \text{ in } H^{-1}(\Omega) \text{ as } n \rightarrow \infty. \quad (4.1)$$

We claim that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. If not, arguing by contradiction, we may assume that $\|u_n\| \rightarrow \infty$ as $n \rightarrow \infty$. Let $w_n = u_n/\|u_n\|$, we see that $\|w_n\| = 1$, and then there exists $w_0 \in H_0^1(\Omega)$ such that as $n \rightarrow \infty$,

$$w_n \rightharpoonup w_0 \text{ in } H_0^1(\Omega), \quad w_n \rightarrow w_0 \text{ in } L^p(\Omega) \quad (p \in (1, 2^*)).$$

We divide into the following two cases to discussion.

Case (i) : $w_0 = 0$.

It follows from (4.1) and $\|u_n\| \rightarrow \infty$ that

$$o(1) = \frac{I(u_n) - \frac{1}{4}I'(u_n)u_n}{\|u_n\|^2} = \frac{a}{4} - \frac{\lambda}{4}\|w_n\|_2^2 = \frac{a}{4} + o(1),$$

this implies $a = 0$, which contradicts $a > 0$.

Case (ii) : $w_0 \neq 0$, i.e., there exists $C_1 > 0$ such that $\|w_n\| \geq \|w_0\| \geq C_1$, up to a subsequence.

Noting that $\|u_n\| \rightarrow \infty$ as $n \rightarrow \infty$, we see that

$$\begin{aligned} o(1) &= \frac{I(u_n)}{\|u_n\|^4} = \frac{a}{2\|u_n\|^2} + \frac{b}{4} - \frac{\lambda\|u_n\|_2^2}{2\|u_n\|^4} - \frac{\delta\|u_n\|_4^4}{4\|u_n\|^4} \\ &= \frac{b}{4} - \frac{\delta}{4}\|w_n\|_4^4 + o(1), \end{aligned}$$

which means that

$$b = \delta\|w_n\|_4^4 + o(1). \quad (4.2)$$

On the other hand, we also obtain that

$$o(1) = \frac{I(u_n) - \frac{1}{4}I'(u_n)u_n}{\|u_n\|^2} = \frac{a}{4} - \frac{\lambda}{4}\|w_n\|_2^2$$

this, together with (4.2), yields that

$$a + o(1) = \lambda\|w_n\|_2^2 \leq \lambda|\Omega|^{\frac{1}{2}}\|w_n\|_4^2 \leq \lambda|\Omega|^{\frac{1}{2}}\left(\frac{b + o(1)}{\delta}\right)^{\frac{1}{2}},$$

so that

$$\delta \leq \frac{b\lambda^2|\Omega|}{a^2},$$

this is a contradiction since $\delta > b\lambda^2|\Omega|/a^2$, therefore our claim holds and $\{u_n\}$ is bounded.

We will complete the proof to show that $\{u_n\}$ has a subsequence which strongly converges in $H_0^1(\Omega)$. In fact, otherwise, by Proposition 2.1, we know that there exist a function $u_0 \in H_0^1(\Omega)$

which is a weak limit of $\{u_n\}$, a number $k \in \mathbb{N}$ and further, for every $i \in \{1, 2, \dots, k\}$, a sequence of values $\{R_n^i\} \subset \mathbb{R}^+$, points $\{x_n^i\} \subset \overline{\Omega}$ and a function $v_i \in \mathcal{D}^{1,2}(\mathbb{R}^4)$ satisfying (2.6), (2.7) and

$$I(u_n) = \tilde{I}(u_0) + \sum_{i=1}^k \tilde{I}^\infty(v_i) + o(1) \quad (4.3)$$

up to subsequences, where $o(1) \rightarrow 0$ as $n \rightarrow \infty$ and \tilde{I} and \tilde{I}^∞ are respectively given by (2.8) and (2.9).

It follows from the Poincaré inequality that

$$\begin{aligned} \tilde{I}(u_0) &= \tilde{I}(u_0) - \frac{1}{4} \left\{ (a + bA) \|u_0\|^2 - \lambda \|u_0\|_2^2 - \delta \|u_0\|_4^4 \right\} \\ &\geq \frac{a}{4} \left(1 - \frac{\lambda}{a\lambda_1} \right) \|u_0\|^2, \end{aligned} \quad (4.4)$$

where $A := \|u_0\|^2 + \sum_{i=1}^k \|v_i\|_{\mathcal{D}}^2$. By (2.9) and the Sobolev inequality we get

$$\begin{aligned} 0 &= (a + bA) \|v_i\|_{\mathcal{D}}^2 - \delta \int_{\mathbb{R}^4} |v_i|^4 dx \\ &\geq (a + b\|u_0\|^2 + b\|v_i\|_{\mathcal{D}}^2) \|v_i\|_{\mathcal{D}}^2 - \frac{\delta \|v_i\|_{\mathcal{D}}^4}{\mathcal{S}^2} \\ &= (a + b\|u_0\|^2) \|v_i\|_{\mathcal{D}}^2 - \frac{\delta - b\mathcal{S}^2}{\mathcal{S}^2} \|v_i\|_{\mathcal{D}}^4, \end{aligned}$$

so that

$$\|v_i\|_{\mathcal{D}}^2 \geq \frac{\mathcal{S}^2(a + b\|u_0\|^2)}{\delta - b\mathcal{S}^2}, \quad (4.5)$$

here we have used the inequality $b\mathcal{S}^2 < \delta$ which can be obtained from the assumptions $\lambda \geq a\lambda_1$, $\delta > b|\Omega|\lambda^2/a^2$ and the following inequality

$$0 < \|\varphi_{11}\|^4 = \lambda_1^2 \|\varphi_{11}\|_2^4 \leq \frac{\lambda^2}{a^2} \|\varphi_{11}\|_2^4 \leq \frac{\lambda^2 |\Omega|}{a^2} \|\varphi_{11}\|_4^4 < \frac{\lambda^2 |\Omega|}{a^2 \mathcal{S}^2} \|\varphi_{11}\|^4, \quad (4.6)$$

where φ_{11} is the positive eigenfunction of the first eigenvalue λ_1 of $(-\Delta, H_0^1(\Omega))$.

From (2.7) and (2.9), we see that for $i \in \{1, 2, \dots, k\}$,

$$\tilde{I}^\infty(v_i) = \tilde{I}^\infty(v_i) - \frac{1}{4} \left\{ (a + bA) \|v_i\|_{\mathcal{D}}^2 - \int_{\mathbb{R}^4} |v_i|^4 dx \right\} \geq \frac{a}{4} \|v_i\|_{\mathcal{D}}^2, \quad (4.7)$$

which, together with (4.3), (4.4), (4.5) and (4.7), implies that

$$\begin{aligned} c &= \lim_{n \rightarrow \infty} I(u_n) = \tilde{I}(u_0) + \sum_{i=1}^k \tilde{I}^\infty(v_i) \\ &\geq \frac{a}{4} \left(1 - \frac{\lambda}{a\lambda_1} \right) \|u_0\|^2 + \frac{(a\mathcal{S})^2}{4(\delta - b\mathcal{S}^2)} + \frac{ab\mathcal{S}^2 \|u_0\|^2}{4(\delta - b\mathcal{S}^2)} \\ &= \frac{(a\mathcal{S})^2}{4(\delta - b\mathcal{S}^2)} + \frac{a}{4} \left(\frac{b\mathcal{S}^2}{\delta - b\mathcal{S}^2} + 1 - \frac{\lambda}{a\lambda_1} \right) \|u_0\|^2 \\ &= \frac{(a\mathcal{S})^2}{4(\delta - b\mathcal{S}^2)} + \frac{a}{4} \left(\frac{\delta}{\delta - b\mathcal{S}^2} - \frac{\lambda}{a\lambda_1} \right) \|u_0\|^2 \\ &\geq \frac{(a\mathcal{S})^2}{4(\delta - b\mathcal{S}^2)} \end{aligned}$$

since $\lambda \leq a\lambda_1\delta/(\delta - b\mathcal{S}^2)$. This contradicts $0 < c < (a\mathcal{S})^2/(4(\delta - b\mathcal{S}^2))$, and then we complete proof of this lemma. \blacksquare

Remark 4.1 Our case of $\lambda \geq a\lambda_1$ is quite different from the case of $\lambda \in (0, a\lambda_1)$ assumed in [25] since, except for the indefiniteness of the operator $-\Delta - a\lambda$, it is not obvious whether $\tilde{I}(u_0) \geq 0$ or not from (4.4), and now it follows from (4.6) that

$$\mathcal{S}^2 < \frac{\lambda^2 |\Omega|}{a^2}, \quad (4.8)$$

which is also different from the assumption given by [25, Lemma 2.1].

To prove Theorems 1.2 and 1.3, we will find two suitable closed subspace V and W with $V \cap W \neq \{0\}$ and $V \oplus W = H$ such that the functional I satisfies the condition $(a_1) - (a_3)$ of Theorem 2.1 with $\beta = (a\mathcal{S})^2/(4(\delta - b\mathcal{S}^2))$.

Given $a, \lambda, \rho > 0$, we set

$$\begin{aligned} \lambda^+ &= \min\{\lambda_j : \lambda < a\lambda_j\}, \quad \lambda^- = \max\{\lambda_j : a\lambda_j \leq \lambda\}, \\ H_1 &= \overline{\oplus_{\lambda_j \geq \lambda^+} M(\lambda_j)}, \quad H_2 = \oplus_{\lambda_j \leq \lambda^-} M(\lambda_j), \quad B_\rho = \{x \in \mathbb{R}^4 : |x| < \rho\}. \end{aligned} \quad (4.9)$$

Without loss of generality, we can suppose that $0 \in \Omega$ and that $B_1(0) \subset \Omega$. Given $\varepsilon > 0$, let $U_\varepsilon(x)$ be the function given by (2.3), we define

$$\Psi_\varepsilon(x) = \phi(x)U_\varepsilon(x),$$

where $\phi \in C_0^\infty(B_1(0))$ with $\phi(x) = 1$ in $B_{1/2}(0)$. Similarly as in [6, 9], we have the following energy estimates of Ψ_ε :

Lemma 4.2

$$\|\Psi_\varepsilon\|^2 = \mathcal{S}^2 + O(\varepsilon); \quad (4.10)$$

$$\|\Psi_\varepsilon\|_4^4 = \mathcal{S}^2 + O(\varepsilon^2); \quad (4.11)$$

$$\|\Psi_\varepsilon\|_2^2 \geq K_1 \varepsilon |\log \varepsilon| + O(\varepsilon), \quad (4.12)$$

$$\|\Psi_\varepsilon\|_1 \leq K_2 \varepsilon^{\frac{1}{2}}, \quad (4.13)$$

$$\|\Psi_\varepsilon\|_3^3 \leq K_3 \varepsilon^{\frac{1}{2}} \quad (4.14)$$

Set $\bar{j} = \max\{j : \lambda_j \leq \lambda\}$ and denote by P_j the orthogonal projector onto the eigenspace $M(\lambda_j)$ corresponding to λ_j . We now set

$$\tilde{\Psi}_\varepsilon = \Psi_\varepsilon - \sum_{j=1}^{\bar{j}} P_j \Psi_\varepsilon \quad (4.15)$$

and give some energy estimates for $\tilde{\Psi}_\varepsilon$ in the following lemma.

Lemma 4.3 *For $\varepsilon > 0$ sufficiently small, we have*

$$\left\| \sum_{j=1}^{\bar{j}} P_j \Psi_\varepsilon \right\|_2^2 \leq C\varepsilon; \quad (4.16)$$

$$\left\| \sum_{j=1}^{\bar{j}} P_j \Psi_\varepsilon \right\|_\infty \leq C\varepsilon^{\frac{1}{2}}; \quad (4.17)$$

$$\frac{a\|\tilde{\Psi}_\varepsilon\|^2 - \lambda\|\tilde{\Psi}_\varepsilon\|_2^2}{\|\tilde{\Psi}_\varepsilon\|_4^2} = a\mathcal{S} + C\varepsilon \log \varepsilon + O(\varepsilon), \quad (4.18)$$

where $C > 0$ denote variant different constants.

Proof. Since $\{\varphi_{ij}\}$ be the orthogonal eigenfunctions corresponding to λ_j spanning $M(\lambda_j)$, $i = 1, 2, \dots, i_j$, where $i_j = \dim M(\lambda_j)$, then

$$\int_{\Omega} \varphi_{ij} \varphi_{kn} dx = \begin{cases} 1, & \text{if } i = k \text{ and } j = n, \\ 0, & \text{others.} \end{cases} \quad (4.19)$$

Since $\Psi_{\varepsilon} \in H_0^1(\Omega)$ and $\{\varphi_{ij}\} (i = 1, 2, \dots, i_j; j = 1, 2, \dots)$ is a family of orthonormal basis of $H_0^1(\Omega)$, there holds

$$\Psi_{\varepsilon}(x) = \sum_{j=1}^{\infty} \sum_{i=1}^{i_j} \left(\int_{\Omega} \Psi_{\varepsilon} \varphi_{ij} dx \right) \varphi_{ij}(x), \quad (4.20)$$

then we have

$$P_j \Psi_{\varepsilon}(x) = \sum_{i=1}^{i_j} \left(\int_{\Omega} \Psi_{\varepsilon} \varphi_{ij} dx \right) \varphi_{ij}(x). \quad (4.21)$$

On the other hand, recalling that $\varphi_{ij} \in C^{\infty}(\Omega)$, we see that there exists $C > 0$, dependent of j , such that $\sup\{\varphi_{ij}(x) : x \in \Omega, i = 1, 2, \dots, i_j\} \leq C$, therefore, by (4.13) and (4.21) we know

$$\|P_j \Psi_{\varepsilon}\|_2^2 = \sum_{i=1}^{i_j} \left(\int_{\Omega} \Psi_{\varepsilon} \varphi_{ij} dx \right)^2 \leq C \left(\sum_{i=1}^{i_j} \|\varphi_{ij}\|_{\infty}^2 \right) \|\Psi_{\varepsilon}\|_1^2 \leq C \|\Psi_{\varepsilon}\|_1^2 \leq C\varepsilon. \quad (4.22)$$

Similarly, we also deduce

$$\begin{aligned} \left\| \sum_{j=1}^{\bar{j}} P_j \Psi_{\varepsilon} \right\|_2^2 &= \sum_{j=1}^{\bar{j}} \|P_j \Psi_{\varepsilon}\|_2^2 \leq C\varepsilon, \\ \|P_j \Psi_{\varepsilon}\|_{\infty} &\leq \sum_{i=1}^{i_j} \left(\int_{\Omega} \Psi_{\varepsilon} \varphi_{ij} dx \right) \|\varphi_{ij}(x)\|_{\infty} \leq C \|\Psi_{\varepsilon}\|_1 \leq C\varepsilon^{\frac{1}{2}}, \\ \left\| \sum_{j=1}^{\bar{j}} P_j \Psi_{\varepsilon} \right\|_{\infty} &= \sum_{j=1}^{\bar{j}} \|P_j \Psi_{\varepsilon}\|_{\infty} \leq C\varepsilon^{\frac{1}{2}}. \end{aligned} \quad (4.23)$$

Thus (4.16) and (4.17) hold. Furthermore, we have, by (4.14) and (4.23),

$$\begin{aligned} \left| \int_{\Omega} (|\tilde{\Psi}_{\varepsilon}|^4 - |\Psi_{\varepsilon}|^4) dx \right| &= 4 \left| \int_{\Omega} \int_0^1 \left| \Psi_{\varepsilon} - t \sum_{j=1}^{\bar{j}} P_j \Psi_{\varepsilon} \right|^2 \left(\Psi_{\varepsilon} - t \sum_{j=1}^{\bar{j}} P_j \Psi_{\varepsilon} \right) \left(\sum_{j=1}^{\bar{j}} P_j \Psi_{\varepsilon} \right) dt dx \right| \\ &\leq 32 \left| \int_{\Omega} \int_0^1 \left(|\Psi_{\varepsilon}|^3 + \left| t \sum_{j=1}^{\bar{j}} P_j \Psi_{\varepsilon} \right|^3 \right) \left(\sum_{j=1}^{\bar{j}} P_j \Psi_{\varepsilon} \right) dt dx \right| \\ &\leq 32 \left(\|\Psi_{\varepsilon}\|_3^3 \left\| \sum_{j=1}^{\bar{j}} P_j \Psi_{\varepsilon} \right\|_{\infty} + \left\| \sum_{j=1}^{\bar{j}} P_j \Psi_{\varepsilon} \right\|_4^4 \right) \\ &\leq 32 \left(\|\Psi_{\varepsilon}\|_3^3 \left\| \sum_{j=1}^{\bar{j}} P_j \Psi_{\varepsilon} \right\|_{\infty} + \left\| \sum_{j=1}^{\bar{j}} P_j \Psi_{\varepsilon} \right\|_2^2 \left\| \sum_{j=1}^{\bar{j}} P_j \Psi_{\varepsilon} \right\|_{\infty}^2 \right) \\ &\leq C(\varepsilon + \varepsilon^2), \end{aligned}$$

which implies that

$$\|\tilde{\Psi}_{\varepsilon}\|_4^4 = \|\Psi_{\varepsilon}\|_4^4 + O(\varepsilon). \quad (4.24)$$

It follows from (4.19), (4.20), (4.21), (4.22) and (4.23) that

$$\begin{aligned}
\|\tilde{\Psi}_\varepsilon\|^2 &= \int_\Omega \left| \nabla \Psi_\varepsilon - \sum_{j=1}^{\bar{j}} \nabla (P_j \Psi_\varepsilon) \right|^2 dx \\
&= \int_\Omega \left\{ |\nabla \Psi_\varepsilon|^2 - 2 \sum_{j=1}^{\bar{j}} \nabla \Psi_\varepsilon \nabla (P_j \Psi_\varepsilon) + \sum_{j=1}^{\bar{j}} |\nabla (P_j \Psi_\varepsilon)|^2 \right\} dx \\
&= \|\Psi_\varepsilon\|^2 - 2 \sum_{j=1}^{\bar{j}} \sum_{i=1}^{i_j} \int_\Omega \Psi_\varepsilon \varphi_{ij} dx \int_\Omega \nabla \Psi_\varepsilon \nabla \varphi_{ij} dx + \sum_{j=1}^{\bar{j}} \sum_{i=1}^{i_j} \left(\int_\Omega \Psi_\varepsilon \varphi_{ij} dx \right)^2 \int_\Omega |\nabla \varphi_{ij}|^2 dx \\
&= \|\Psi_\varepsilon\|^2 - \sum_{j=1}^{\bar{j}} \sum_{i=1}^{i_j} \lambda_j \left(\int_\Omega \Psi_\varepsilon \varphi_{ij} dx \right)^2 \\
&= \|\Psi_\varepsilon\|^2 - \sum_{j=1}^{\bar{j}} \lambda_j \|P_j \Psi_\varepsilon\|_2^2,
\end{aligned} \tag{4.25}$$

$$\begin{aligned}
\|\tilde{\Psi}_\varepsilon\|_2^2 &= \int_\Omega \left| \Psi_\varepsilon - \sum_{j=1}^{\bar{j}} (P_j \Psi_\varepsilon) \right|^2 dx \\
&= \int_\Omega \left\{ |\Psi_\varepsilon|^2 - 2 \sum_{j=1}^{\bar{j}} \Psi_\varepsilon (P_j \Psi_\varepsilon) + \sum_{j=1}^{\bar{j}} |P_j \Psi_\varepsilon|^2 \right\} dx \\
&= \|\Psi_\varepsilon\|_2^2 - 2 \sum_{j=1}^{\bar{j}} \sum_{i=1}^{i_j} \left(\int_\Omega \Psi_\varepsilon \varphi_{ij} dx \right)^2 + \sum_{j=1}^{\bar{j}} \sum_{i=1}^{i_j} \left(\int_\Omega \Psi_\varepsilon \varphi_{ij} dx \right)^2 \int_\Omega |\varphi_{ij}|^2 dx \\
&= \|\Psi_\varepsilon\|_2^2 - \sum_{j=1}^{\bar{j}} \sum_{i=1}^{i_j} \left(\int_\Omega \Psi_\varepsilon \varphi_{ij} dx \right)^2 \\
&= \|\Psi_\varepsilon\|_2^2 - \sum_{j=1}^{\bar{j}} \|P_j \Psi_\varepsilon\|_2^2.
\end{aligned} \tag{4.26}$$

Combining (4.24), (4.25), (4.26) and (4.10)-(4.12) of Lemma 4.2, we obtain that

$$\begin{aligned}
\frac{a\|\tilde{\Psi}_\varepsilon\|^2 - \lambda\|\tilde{\Psi}_\varepsilon\|_2^2}{\|\tilde{\Psi}_\varepsilon\|_4^2} &= \frac{a\|\Psi_\varepsilon\|^2 - \lambda\|\Psi_\varepsilon\|_2^2 + \sum_{j=1}^{\bar{j}} (\lambda - a\lambda_j) \|P_j \Psi_\varepsilon\|_2^2}{\|\Psi_\varepsilon\|_4^2 + O(\varepsilon)} \\
&= \frac{a\|\Psi_\varepsilon\|^2 - \lambda\|\Psi_\varepsilon\|_2^2 + O(\varepsilon)}{\|\Psi_\varepsilon\|_4^2 + O(\varepsilon^2)} \\
&= \frac{a\|\Psi_\varepsilon\|^2 - \lambda\|\Psi_\varepsilon\|_2^2}{\|\Psi_\varepsilon\|_4^2} + O(\varepsilon) \\
&= a\mathcal{S} + C\varepsilon \log \varepsilon + O(\varepsilon),
\end{aligned}$$

then (4.18) is proved and this completes the proof of the lemma. \blacksquare

Let H_1 and H_2 be given in (4.9). Clearly $\tilde{\Psi}_\varepsilon$ defined by (4.15) belongs to H_1 . Now we define a subspace \overline{W}_ε in $H_0^1(\Omega)$ as

$$\overline{W}_\varepsilon = \{u \in H_0^1(\Omega) : u = \bar{u} + t\tilde{\Psi}_\varepsilon, \bar{u} \in H_2, t \in \mathbb{R}\}, \tag{4.27}$$

then we have the following result:

Lemma 4.4 Suppose that $a, \lambda > 0$, then for $\varepsilon > 0$ sufficiently small,

$$\sup_{u \in \overline{W}_\varepsilon \setminus \{0\}} \frac{\|u\|^4}{\|u\|_4^4} \leq 2|\Omega| \left(\sum_{j=1}^{\bar{j}} i_j \right)^2 \frac{\lambda^2}{a^2}.$$

Proof. By the definition of \overline{W}_ε and the orthogonality of H_1 and H_2 , we know that for each $u \in \overline{W}_\varepsilon$, $u = \overline{u} + t\tilde{\Psi}_\varepsilon$ for some $\overline{u} \in H_2$ and $t \in \mathbb{R}$, and

$$\|u\|^4 = \|\overline{u} + t\tilde{\Psi}_\varepsilon\|^4 = (\|\overline{u} + t\tilde{\Psi}_\varepsilon\|^2)^2 \leq 2\|\overline{u}\|^4 + 2\|t\tilde{\Psi}_\varepsilon\|^4. \quad (4.28)$$

Since

$$\|\overline{u}\|_\infty \leq C, \quad \|\tilde{\Psi}_\varepsilon\|_\infty = \|\Psi_\varepsilon - \sum_{j=1}^{\bar{j}} P_j \Psi_\varepsilon\|_\infty \leq \|\Psi_\varepsilon\|_\infty + \left\| \sum_{j=1}^{\bar{j}} P_j \Psi_\varepsilon \right\|_\infty \leq 8\varepsilon^{-\frac{1}{2}} + C\varepsilon^{\frac{1}{2}},$$

we have

$$\left| \int_\Omega \overline{u}(t\tilde{\Psi}_\varepsilon)^3 dx \right| \leq |t|^3 \|\tilde{\Psi}_\varepsilon\|_\infty \left| \int_\Omega \overline{u} \tilde{\Psi}_\varepsilon dx \right| = 0, \quad \int_\Omega \overline{u}^3 (t\tilde{\Psi}_\varepsilon) dx = \int_\Omega \overline{u}^2 (t\tilde{\Psi}_\varepsilon)^2 dx = 0,$$

so that

$$\begin{aligned} \|u\|_4^4 &= \int_\Omega |\overline{u} + t\tilde{\Psi}_\varepsilon|^4 dx \\ &= \int_\Omega (|\overline{u}|^4 + 4\overline{u}^3(t\tilde{\Psi}_\varepsilon) + 6\overline{u}^2(t\tilde{\Psi}_\varepsilon)^2 + 4\overline{u}(t\tilde{\Psi}_\varepsilon)^3 + |t\tilde{\Psi}_\varepsilon|^4) dx \\ &= \|\overline{u}\|_4^4 + \|t\tilde{\Psi}_\varepsilon\|_4^4. \end{aligned} \quad (4.29)$$

Therefore, by using (4.28), (4.29) and Lemma 2.1, we see that

$$\begin{aligned} \sup_{u \in \overline{W}_\varepsilon \setminus \{0\}} \frac{\|u\|^4}{\|u\|_4^4} &\leq \sup_{u \in \overline{W}_\varepsilon \setminus \{0\}} \frac{2\|\overline{u}\|^4 + 2\|t\tilde{\Psi}_\varepsilon\|^4}{\|\overline{u}\|_4^4 + \|t\tilde{\Psi}_\varepsilon\|_4^4} \\ &\leq 2 \max \left\{ \sup_{\overline{u} \in H_2 \setminus \{0\}} \frac{\|\overline{u}\|^4}{\|\overline{u}\|_4^4}, \frac{\|t\tilde{\Psi}_\varepsilon\|^4}{\|t\tilde{\Psi}_\varepsilon\|_4^4} \right\} \\ &= 2 \max \left\{ \sup_{\overline{u} \in H_2 \setminus \{0\}} \frac{\|\overline{u}\|^4}{\|\overline{u}\|_4^4}, \frac{\mathcal{S}^4 + O(\varepsilon)}{\mathcal{S}^2 + O(\varepsilon^2)} \right\}. \end{aligned} \quad (4.30)$$

On the other hand, for each $\overline{u} \in H_2 \setminus \{0\}$, there exists $\{c_{ij}\} \subset \mathbb{R}$ ($j = 1, 2, \dots, \bar{j}$; $i = 1, 2, \dots, i_j$) such that

$$\overline{u} = \sum_{j=1}^{\bar{j}} \sum_{i=1}^{i_j} c_{ij} \varphi_{ij},$$

where $\{\varphi_{ij}\}$ is the family of orthonormal basis of $H_0^1(\Omega)$ given in the proof of Lemma 4.3. Therefore we have

$$\begin{aligned} \|\overline{u}\|^4 &= \left(\sum_{j=1}^{\bar{j}} \sum_{i=1}^{i_j} \|c_{ij} \varphi_{ij}\|^2 \right)^2 \leq \left(\sum_{j=1}^{\bar{j}} i_j \right)^2 \left(\sum_{j=1}^{\bar{j}} \sum_{i=1}^{i_j} \|c_{ij} \varphi_{ij}\|^4 \right), \\ \|\overline{u}\|_4^4 &= \sum_{j=1}^{\bar{j}} \sum_{i=1}^{i_j} \|c_{ij} \varphi_{ij}\|_4^4. \end{aligned}$$

By using Lemma 2.1 again, we get

$$\sup_{\overline{u} \in H_2 \setminus \{0\}} \frac{\|\overline{u}\|^4}{\|\overline{u}\|_4^4} \leq \left(\sum_{j=1}^{\bar{j}} i_j \right)^2 \max \left\{ \frac{\|\varphi_{ij}\|^4}{\|\varphi_{ij}\|_4^4}, j = 1, 2, \dots, \bar{j}, \quad i = 1, 2, \dots, i_j \right\}. \quad (4.31)$$

It follows from (4.6), (4.30), (4.31) and (4.8) in Remark 4.1 that

$$\begin{aligned}
\sup_{u \in \overline{W}_\varepsilon \setminus \{0\}} \frac{\|u\|^4}{\|u\|_4^4} &\leq 2 \max \left\{ \frac{\mathcal{S}^4 + O(\varepsilon)}{\mathcal{S}^2 + O(\varepsilon^2)}, \left(\sum_{j=1}^{\bar{j}} i_j \right)^2 \frac{\|\varphi_{ij}\|^4}{\|\varphi_{ij}\|_4^4}, \quad j = 1, 2, \dots, \bar{j}, \quad i = 1, 2, \dots, i_j \right\} \\
&\leq 2 \max \left\{ \frac{\mathcal{S}^4 + O(\varepsilon)}{\mathcal{S}^2 + O(\varepsilon^2)}, \left(\sum_{j=1}^{\bar{j}} i_j \right)^2 (\lambda_j)^2 |\Omega|, \quad j = 1, 2, \dots, \bar{j} \right\} \\
&\leq 2 |\Omega| \left(\sum_{j=1}^{\bar{j}} i_j \right)^2 \frac{\lambda^2}{a^2}
\end{aligned}$$

for $\varepsilon > 0$ sufficiently small. \blacksquare

Lemma 4.5 *Suppose that $a > 0$ and $\lambda \in [a\lambda_1, a\lambda_1\delta/(\delta - b\mathcal{S}^2))$, then for $\varepsilon > 0$ small,*

$$\sup_{u \in \overline{W}_\varepsilon} I(u) < \frac{(a\mathcal{S})^2}{4(\delta - b\mathcal{S}^2)}. \quad (4.32)$$

provided one of the following conditions holds

- (1) $\delta > 0$, $b > 0$ is sufficiently small;
- (2) $b > 0$, $\delta > 0$ is sufficiently large.

Proof. First of all, let $\delta > 0$ be fixed, we may choose $b_1 > 0$ such that for each $b \in (0, b_1)$, it holds

$$\frac{a\lambda_1\delta}{\delta - b\mathcal{S}^2} \leq a\lambda_2, \quad 4b\lambda_2|\Omega| < \delta, \quad (4.33)$$

thus $H_2 = M(\lambda_1)$. It follows from Lemma 4.4 that there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$

$$\sup_{u \in \overline{W}_\varepsilon \setminus \{0\}} \frac{\|u\|^4}{\|u\|_4^4} \leq 2\lambda_2^2|\Omega|.$$

For each $u \in \overline{W}_\varepsilon$, we may assume that $a\|u\|^2 - \lambda\|u\|_2^2 > 0$ (since otherwise $I(tu) \leq 0$ for all $t > 0$, which implies $\sup_{u \in \overline{W}_\varepsilon} I(u) \leq \sup_{u \in \overline{W}_\varepsilon} \max_{t>0} I(tu) \leq 0$, so that the desired inequality (4.32) holds trivially), hence $I(tu) > 0$ for $t > 0$ small, and

$$0 < \max_{t>0} I(tu) \leq \frac{(a\|u\|^2 - \lambda\|u\|_2^2)^2}{4(\delta - b\frac{\|u\|_4^4}{\|u\|_4^4})\|u\|_4^4} \leq \frac{(a\|u\|^2 - \lambda\|u\|_2^2)^2}{4(\delta - 2b\lambda_2^2|\Omega|)\|u\|_4^4}. \quad (4.34)$$

Next, we claim that

$$\sup\{a\|u\|^2 - \lambda\|u\|_2^2 : u \in \overline{W}_\varepsilon, \|u\|_4 = 1\} \leq a\mathcal{S} + C\varepsilon \log \varepsilon + O(\varepsilon)$$

holds for some positive constant C and small ε .

Indeed, for $u \in \overline{W}_\varepsilon$ with $\|u\|_4 = 1$, we have

$$1 = \|u\|_4^4 = \|\overline{u}\|_4^4 + \|t\tilde{\Psi}_\varepsilon\|_4^4,$$

then by (4.24), we see that t is bounded and $\|t\tilde{\Psi}_\varepsilon\|_4^4 \leq 1$. Therefore, by using Lemma 4.3,

$$\begin{aligned}
a\|u\|^2 - \lambda\|u\|_2^2 &= a(\|\bar{u}\|^2 + \|t\tilde{\Psi}_\varepsilon\|_2^2) - \lambda(\|\bar{u}\|_2^2 + \|t\tilde{\Psi}_\varepsilon\|_2^2) \\
&= a\|\bar{u}\|^2 - \lambda\|\bar{u}\|_2^2 + a\|t\tilde{\Psi}_\varepsilon\|^2 - \lambda\|t\tilde{\Psi}_\varepsilon\|_2^2 \\
&= (a\lambda_1 - \lambda)\|\bar{u}\|_2^2 + \frac{a\|\tilde{\Psi}_\varepsilon\|^2 - \lambda\|\tilde{\Psi}_\varepsilon\|_2^2}{\|\tilde{\Psi}_\varepsilon\|_4^2} \|t\tilde{\Psi}_\varepsilon\|_4^2 \\
&\leq \frac{a\|\tilde{\Psi}_\varepsilon\|^2 - \lambda\|\tilde{\Psi}_\varepsilon\|_2^2}{\|\tilde{\Psi}_\varepsilon\|_4^2} \\
&\leq a\mathcal{S} + C\varepsilon \log \varepsilon + O(\varepsilon).
\end{aligned} \tag{4.35}$$

It follows from (4.34) and (4.35) that

$$\begin{aligned}
\sup_{u \in \bar{W}_\varepsilon} I(u) &\leq \frac{(a\mathcal{S})^2 + C\varepsilon \log \varepsilon + O(\varepsilon)}{4(\delta - 2b\lambda_2^2|\Omega|)} \\
&= \frac{a^2}{4} \left(\frac{\mathcal{S}^2}{\delta - b\mathcal{S}^2} + \frac{b\mathcal{S}^2(2\lambda_2^2|\Omega| - \mathcal{S}^2)}{(\delta - b\mathcal{S}^2)(\delta - 2b\lambda_2^2|\Omega|)} + \frac{C\varepsilon \log \varepsilon + O(\varepsilon)}{\delta - 2b\lambda_2^2|\Omega|} \right) \\
&\leq \frac{a^2}{4} \left(\frac{\mathcal{S}^2}{\delta - b\mathcal{S}^2} + \frac{4b\mathcal{S}^2(2\lambda_2^2|\Omega| - \mathcal{S}^2)}{\delta^2} + \frac{C\varepsilon \log \varepsilon + O(\varepsilon)}{\delta} \right),
\end{aligned}$$

so that there exists $\varepsilon_1 > 0$ such that for each $\varepsilon \in (0, \varepsilon_1)$, we can find $b_0 := b_0(\varepsilon)$ satisfying that for each $b \in (0, b_0)$ the inequality (4.32) remains true.

Finally, if $b > 0$ is fixed then (4.33) obviously holds for all large $\delta > 0$. Similarly as in the above, we see that (4.32) also holds for all large $\delta > 0$. Thus we have proved this lemma. \blacksquare

Remark 4.2 In Lemma 4.5, we have shown that the assumption (a_{31}) of Theorem 2.1 holds with $W = \bar{W}_\varepsilon$ and $\beta = (a\mathcal{S})^2/(4(\delta - b\mathcal{S}^2))$, the choice of β is consistent with Lemma 4.1. However, by checking the proof of Lemma 4.5 carefully, we observe that the conclusion of Lemma 4.5 is still true if we take the place of $(a\mathcal{S})^2/(4(\delta - b\mathcal{S}^2))$ by $(a\mathcal{S})^2/(4\delta) - \sigma$ for some $\sigma > 0$ small. Indeed, it follows from (4.34) and (4.35) that

$$\begin{aligned}
\sup_{u \in \bar{W}_\varepsilon} I(u) &\leq \frac{(a\mathcal{S})^2 + C\varepsilon \log \varepsilon + O(\varepsilon)}{4(\delta - 2b\lambda_2^2|\Omega|)} \\
&= \frac{a^2}{4} \left(\frac{\mathcal{S}^2}{\delta} + \frac{2b\mathcal{S}^2\lambda_2^2|\Omega|}{\delta(\delta - 2b\lambda_2^2|\Omega|)} + \frac{C\varepsilon \log \varepsilon + O(\varepsilon)}{\delta - 2b\lambda_2^2|\Omega|} \right) \\
&\leq \frac{a^2}{4} \left(\frac{\mathcal{S}^2}{\delta} + \frac{4b\mathcal{S}^2\lambda_2^2|\Omega|}{\delta^2} + \frac{C\varepsilon \log \varepsilon + O(\varepsilon)}{\delta} \right) \\
&< \frac{(a\mathcal{S})^2}{4\delta} - \sigma
\end{aligned}$$

since for ε sufficiently small, we can take $b \ll \varepsilon$ or δ large enough.

Lemma 4.6 Suppose that $a, \delta > 0$. Let $\{b_n\}$ and $\{\bar{\lambda}_n\}$ be two sequences of positive numbers satisfying $b_n \rightarrow 0$ and $\bar{\lambda}_n \rightarrow a\lambda_1$ as $n \rightarrow \infty$, where $\bar{\lambda}_n > a\lambda_1$ for all $n \in \mathbb{N}$. If u_n is the solution of Eq. (1.3) corresponding to b_n and $\bar{\lambda}_n$ and $\{I(u_n)\}$ is bounded, then $\{u_n\}$ is bounded.

Proof. Arguing by contradiction, we may assume $\|u_n\| \rightarrow \infty$ as $n \rightarrow \infty$. Let $w_n = u_n/\|u_n\|$, it is easy to see that $\|w_n\| = 1$, and then there exists $w_0 \in H_0^1(\Omega)$ such that $w_n \rightharpoonup w_0$ in $H_0^1(\Omega)$ and

$w_n \rightarrow w_0$ in $L^p(\Omega)$ ($1 < p < 4$) as $n \rightarrow \infty$, thus we have

$$0 = \frac{I'(u_n)u_n}{\|u_n\|^4} = \frac{a}{\|u_n\|^2} + b_n - \frac{\bar{\lambda}_n \|w_n\|_2^2}{\|u_n\|^2} - \delta \|w_n\|_4^4,$$

which implies that $\|w_n\|_4^4 \rightarrow 0$ as $n \rightarrow \infty$, therefore we obtain $w_0 = 0$. On the other hand,

$$o(1) = \frac{I(u_n) - \frac{1}{4}I'(u_n)u_n}{\|u_n\|^2} = \frac{a}{4} - \frac{\bar{\lambda}_n}{4} \|w_n\|_2^2 = \frac{a}{4} + o(1),$$

we obtain $a = 0$, this is a contradiction since $a > 0$, therefore $\{u_n\}$ is bounded in $H_0^1(\Omega)$. \blacksquare

Lemma 4.7 Suppose that $a > 0$, $b > 0$ and $\lambda > a\lambda_1$. Let $0 < \delta < b\mathcal{S}^2$. Then Eq. (1.3) has a positive ground state solution $u \in H_0^1(\Omega)$ with $I(u) < 0$.

Proof. Let φ_{11} be the positive eigenfunction corresponding to the first eigenvalue λ_1 , then we see that

$$a\|\varphi_{11}\|^2 - \lambda\|\varphi_{11}\|_2^2 = (a - \frac{\lambda}{\lambda_1})\|\varphi_{11}\|^2 < 0 \quad \text{and} \quad b\|\varphi_{11}\|^4 - \delta\|\varphi_{11}\|_4^4 > (b - \frac{\delta}{\mathcal{S}^2})\|\varphi_{11}\|^4 \geq 0$$

since $\lambda > a\lambda_1$ and $0 < \delta \leq b\mathcal{S}^2$. Therefore,

$$\begin{aligned} I(t\varphi_{11}) &= \frac{a}{2}\|t\varphi_{11}\|^2 + \frac{b}{4}\|t\varphi_{11}\|^4 - \frac{\lambda}{2}\|t\varphi_{11}\|_2^2 - \frac{\delta}{4}\|t\varphi_{11}\|_4^4 \\ &= \frac{1}{2}(a - \frac{\lambda}{\lambda_1})\|\varphi_{11}\|^2 t^2 + \frac{1}{4}(b\|\varphi_{11}\|^4 - \delta\|\varphi_{11}\|_4^4)t^4 < 0 \end{aligned}$$

for $t > 0$ small, i.e., there exists $t_0 > 0$ small such that $I(t_0\varphi_{11}) < 0$. On the other hand,

$$\begin{aligned} I(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\lambda}{2}\|u\|_2^2 - \frac{\delta}{4}\|u\|_4^4 \\ &> (a - \frac{\lambda}{\lambda_1})\|u\|^2 + (b - \frac{\delta}{\mathcal{S}^2})\|u\|^4 \rightarrow +\infty \end{aligned}$$

as $\|u\| \rightarrow \infty$, which means that I is coercive. Let $c = \inf\{I(u) : u \in H_0^1(\Omega)\}$. By the Ekeland variational principle (cf. [32, Theorem 2.4]), there exists $(PS)_c$ sequence $\{u_n\}$ of I . Clearly, $-\infty < c < 0$ and $\{u_n\}$ is bounded. Then by Proposition 2.1 and (2.5), there exists $u \in H_0^1(\Omega) \setminus \{0\}$ such that $u_n \rightarrow u$ in $H_0^1(\Omega)$ as $n \rightarrow \infty$, up to a subsequence. Therefore, u is a ground state solution of Eq. (1.3) satisfying $I(u) = c < 0$. Noting that $I(|u|) = I(u)$, so that $|u|$ is also a global minimizer of I on $H_0^1(\Omega)$, in this way we have that $|u|$ is a ground state solution, then $|u| > 0$ because of the maximum principle, therefore we obtain a positive ground state solution. \blacksquare

Lemma 4.8 Assume that $a > 0$, $b > 0$ and $0 < \delta < b\mathcal{S}^2$ with $\lambda > a\lambda_1$. Let $\{\delta_n\}$ be a decreasing sequence satisfying $\delta_n \rightarrow 0$ as $n \rightarrow \infty$ and let u_n be the positive ground state solution corresponding to δ_n , then $\{u_n\}$ is bounded.

Proof. Arguing by contradiction, we assume $\|u_n\| \rightarrow \infty$ as $n \rightarrow \infty$. Set $w_n = u_n/\|u_n\|$, then $\|w_n\| = 1$ and there exists $w_0 \in H_0^1(\Omega)$ such that $w_n \rightharpoonup w_0$ in $H_0^1(\Omega)$ and $w_n \rightarrow w_0$ in $L^p(\Omega)$ ($1 < p < 4$) as $n \rightarrow \infty$. Clearly, $\|w_0\| \leq 1$. Denote $c_n := I(u_n)$. It follows from $I'(u_n)u_n = 0$ that

$$\lim_{n \rightarrow \infty} \frac{c_n}{\|u_n\|^4} = \lim_{n \rightarrow \infty} \frac{I(u_n) - \frac{1}{2}I'(u_n)u_n}{\|u_n\|^4} = \lim_{n \rightarrow \infty} \frac{\delta_n}{4} \|w_n\|_4^4 - \frac{b}{4} = -\frac{b}{4} < 0.$$

On the other hand,

$$\lim_{n \rightarrow \infty} \frac{c_n}{\|u_n\|^2} = \lim_{n \rightarrow \infty} \frac{I(u_n) - \frac{1}{4}I'(u_n)u_n}{\|u_n\|^2} = \frac{a}{4} - \lim_{n \rightarrow \infty} \frac{\lambda}{4}\|w_n\|_2^2 = \frac{a}{4} - \frac{\lambda}{4}\|w_0\|_2^2.$$

Thus we get a contradiction since $\|u_n\| \rightarrow \infty$ as $n \rightarrow \infty$. Therefore, $\{u_n\}$ is bounded. \blacksquare

Lemma 4.9 *Under the assumptions of Lemma 4.8 and let $c_n := I(u_n)$, then $c_n \rightarrow d$ as $n \rightarrow \infty$, where $d := \inf\{J(u) : u \in H_0^1(\Omega)\} < 0$, $J(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\lambda}{2}\|u\|_2^2$ is given by (3.1).*

Proof. By Lemma 4.8, we see that $\{u_n\}$ is bounded, then $\{c_n\}$ is also bounded. Clearly, $\{c_n\}$ is nondecreasing on n , thus $c_n \rightarrow c_0$ for some $c_0 \in \mathbb{R}$ as $n \rightarrow \infty$. Obviously, the functional J is coercive, weakly low semi-continuous and

$$J(t\varphi_{11}) = \frac{1}{2}(a - \frac{\lambda}{\lambda_1})\|\varphi_{11}\|^2 t^2 + \frac{b}{4}\|\varphi_{11}\|^4 t^4 < 0$$

for $t > 0$ small since $\lambda > a\lambda_1$. Therefore, there exists $v \in H_0^1(\Omega) \setminus \{0\}$ such that

$$d = J(v) = \inf\{J(u) : u \in H_0^1(\Omega)\} < 0.$$

It follows from Lemma 4.7 that $c_n \leq d$ since $I(u) \leq J(u)$ for each $u \in H_0^1(\Omega)$ and $\delta_n > 0$, hence we get $c_0 \leq d$. If $c_0 < d$, then there exists $n_0 \in \mathbb{N}$ such that for each $n > n_0$,

$$d \leq J(u_n) = I(u_n) + \frac{\delta_n}{4}\|u_n\|_4^4 = c_n + \frac{\delta_n}{4}\|u_n\|_4^4 \leq c_0 + \frac{d - c_0}{2} = \frac{c_0 + d}{2} < d$$

since $\delta_n \rightarrow 0$ and $\{u_n\}$ is bounded, this is a contradiction. Therefore, we must have $c_0 = d$. \blacksquare

Proof of Theorem 1.2:

Suppose that $0 < \delta \leq b\mathcal{S}^2$. If Eq. (1.3) has a nontrivial solution u for each $\lambda \in (0, a\lambda_1]$ then

$$\begin{aligned} 0 &= I'(u)u = a\|u\|^2 + b\|u\|^4 - \lambda\|u\|_2^2 - \delta\|u\|_4^4 \\ &> (a - \frac{\lambda}{\lambda_1})\|u\|^2 + (b - \frac{\delta}{\mathcal{S}^2})\|u\|^4 \geq 0, \end{aligned}$$

this is impossible, therefore, Eq. (1.3) has no nontrivial solution for $\lambda \in (0, a\lambda_1]$.

Now we assume that $a > 0$, $b > 0$, $\lambda > a\lambda_1$ and $0 < \delta < b\mathcal{S}^2$. Then it follows from Lemma 4.7 that Eq. (1.3) has a positive ground state solution.

Assume that $\lambda \in (a\lambda_k, a\lambda_{k+1}]$ with $k \geq 1$. Let $\{\delta_n\}$ be a decreasing sequence satisfying $\delta_n \rightarrow 0$ as $n \rightarrow \infty$. Let u_n be the positive ground state solution corresponding to δ_n . Lemmas 4.8 and 4.9 show that

$$J(u_n) = I(u_n) + \frac{\delta_n}{4}\|u_n\|_4^4 = c_n + \frac{\delta_n}{4}\|u_n\|_4^4 \rightarrow c_0 = d$$

as $n \rightarrow \infty$, i.e., $\{u_n\}$ is a minimizing sequence of J . Then there exists $0 \leq \tilde{u}_0 \in H_0^1(\Omega)$ such that $u_n \rightharpoonup \tilde{u}_0$ in $H_0^1(\Omega)$ and $u_n \rightarrow \tilde{u}_0$ in $L^p(\Omega)$ ($1 < p < 4$) as $n \rightarrow \infty$. Therefore, we have

$$d \leq J(\tilde{u}_0) \leq \lim_{n \rightarrow \infty} J(u_n) = d,$$

i.e., \tilde{u}_0 is a global minimum of J and then \tilde{u}_0 is a positive ground state solution of (1.4) because of the maximum principle. It follows from Step 1 in the proof of Theorem 1.1 that $\tilde{u}_0 = \bar{u}_1$, where \bar{u}_1 is given by (3.2) with $j = 1$. \blacksquare

Proof of Theorem 1.3:

(i) We set $V = H_1$ and $W = \overline{W}_\varepsilon$ (see (4.27)) with ε so small that (4.32) holds. Clearly, the assumptions (a_1) and (a_{33}) of Theorem 2.1 are satisfied. It follows from Lemmas 4.1 and 4.5 that (a_2) and (a_{31}) hold with $\beta = (a\mathcal{S})^2/(4(\delta - b\mathcal{S}^2))$. If $u \in V$ such that $\|u\| = \rho$ with $\rho > 0$ small enough, then

$$\begin{aligned} I(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\lambda}{2}\|u\|_2^2 - \frac{\delta}{4}\|u\|_4^4 \\ &\geq \frac{a}{2}\left(1 - \frac{\lambda}{a\lambda^+}\right)\|u\|^2 - \frac{\delta}{4\mathcal{S}^2}\|u\|^4 \\ &\geq \kappa_0 > 0 \end{aligned}$$

for some positive constant κ_0 , that is, the assumption (a_{32}) of Theorem 2.1 is also satisfied. Since $\dim(V \cap W) = 1$ and $V \oplus W = H_0^1(\Omega)$, by Theorem 2.1, we get that Eq. (1.3) has one pair of nontrivial solutions with the corresponding functional energy in $[\kappa_0, (a\mathcal{S})^2/(4(\delta - b\mathcal{S}^2))]$.

We claim that the solutions of Eq. (1.3) obtained by the above are all sign-changing.

Indeed, otherwise, for the case of (1), we may assume that there exist $\{b_n\}, \{\bar{\lambda}_n\}$ with $b_n > 0, \bar{\lambda}_n \geq a\lambda_1, b_n \rightarrow 0, \bar{\lambda}_n \rightarrow a\lambda_1$ as $n \rightarrow \infty$ and the corresponding solutions $\{u_n\}$ with $u_n \geq 0$ for all $n \in \mathbb{N}$. By using Lemma 4.6, we see that $\{u_n\}$ is bounded. Recalling that the energy functional of (1.6) is defined as

$$I_0(u) = \frac{a}{2}\|u\|^2 - \frac{a\lambda_1}{2}\|u\|_2^2 - \frac{\delta}{4}\|u\|_4^4, \quad u \in H_0^1(\Omega),$$

we deduce from Theorem 2.1 and Remark 4.2 that for n large enough,

$$0 < \frac{\kappa_0}{2} \leq I_0(u_n) = I(u_n) - \frac{b_n}{4}\|u_n\|^4 + \frac{\bar{\lambda}_n - a\lambda_1}{2}\|u_n\|_2^2 \leq \frac{(a\mathcal{S})^2}{4\delta} - \sigma$$

which, together with Lemma 4.6, implies that $I_0(u_n) \rightarrow c_0 \in [\kappa_0/2, (a\mathcal{S})^2/(4\delta)]$, up to a subsequence. On the other hand, for each $v \in H_0^1(\Omega)$,

$$I'_0(u_n)v = I'(u_n)v - b_n\|u_n\|^2 \int_{\Omega} \nabla u_n \nabla v dx - (\bar{\lambda}_n - a\lambda_1) \int_{\Omega} u_n v dx = o(1),$$

$$I'_0(u_n)u_n = I'(u_n)u_n - b_n\|u_n\|^4 - (\bar{\lambda}_n - a\lambda_1) \int_{\Omega} u_n^2 dx = o(1),$$

therefore, by using the standard arguments in [6, 28, 32], we obtain that there exists $u_0 \in H_0^1(\Omega) \setminus \{0\}$ such that $u_n \rightarrow u_0$ in $H_0^1(\Omega)$ as $n \rightarrow \infty$, up to a subsequence.

Let φ_{11} be the positive eigenfunction of the first eigenvalue λ_1 , then we have

$$\begin{aligned} 0 &= (a + b_n\|u_n\|^2) \int_{\Omega} \nabla u_n \nabla \varphi_{11} dx - \bar{\lambda}_n \int_{\Omega} u_n \varphi_{11} dx - \delta \int_{\Omega} |u_n|^2 u_n \varphi_{11} dx \\ &= b_n \lambda_1 \|u_n\|^2 \int_{\Omega} u_n \varphi_{11} dx - (\bar{\lambda}_n - a\lambda_1) \int_{\Omega} u_n \varphi_{11} dx - \delta \int_{\Omega} |u_n|^2 u_n \varphi_{11} dx \\ &= (b_n \lambda_1 \|u_n\|^2 - (\bar{\lambda}_n - a\lambda_1)) \int_{\Omega} |u_n| \varphi_{11} dx - \delta \int_{\Omega} |u_n|^3 \varphi_{11} dx \\ &\leq o(1) - \delta \int_{\Omega} |u_0|^3 \varphi_{11} dx < 0, \end{aligned}$$

this is a contradiction, so that our claim is true for the case of (1). For the case (2), we observe that if u is a solution of (1.3), then $v = \delta^{1/2}u$ is a solution of

$$\begin{cases} -(a + \frac{b}{\delta} \int_{\Omega} |\nabla v|^2 dx) \Delta v = \lambda v + |v|^2 v, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega, \end{cases}$$

then by using similar arguments as above, we can show that v is sign-changing if δ large enough, so that u is also sign-changing.

(ii) We now prove the asymptotical behavior of the solutions as $b \rightarrow 0$. Fix $a, \delta > 0$. Let $\{b_n\}$ and $\{\bar{\lambda}_n\}$ be two sequences satisfying $b_n \rightarrow 0$ and $\bar{\lambda}_n \rightarrow a\lambda_1$ as $n \rightarrow \infty$ with $\bar{\lambda}_n \geq a\lambda_1$. Let u_n be the solution corresponding to b_n and $\bar{\lambda}_n$ obtained above, it follows from Lemma 4.6 that $\{u_n\}$ is bounded. Similarly as in the proof of the above claim, we see that $u_n \rightarrow u_0 \neq 0$ in $H_0^1(\Omega)$ as $n \rightarrow \infty$ and $I'_0(u_0)v = 0$ for all $v \in H_0^1(\Omega)$, that is, u_0 is a nontrivial solution of Eq. (1.6). A standard argument (cf. [6]) shows that u_0 is a sign-changing solution of Eq. (1.6). \blacksquare

5 Proof of Theorems 1.4 and 1.5

In this section, we are investigating the existence of solutions of Eq. (1.1) for $\lambda > a\lambda_1$ and $2 < q < 4$. We regard Eq. (1.1) as a perturbed equation of the following equation

$$\begin{cases} -a\Delta u = \lambda|u|^{q-2}u + \delta|u|^2u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (5.1)$$

then the corresponding functionals for Eq (1.1) and Eq. (5.1) can be respectively written as

$$\bar{I}_b(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\lambda}{q} \int_{\Omega} |u|^q dx - \frac{\delta}{4} \int_{\Omega} |u|^4 dx,$$

and

$$\bar{I}_0(u) = \frac{a}{2}\|u\|^2 - \frac{\lambda}{q} \int_{\Omega} |u|^q dx - \frac{\delta}{4} \int_{\Omega} |u|^4 dx.$$

It follows from [6] (see also [28]) that Eq. (5.1) has a positive ground state solution $u_0 \in H_0^1(\Omega)$, thus we obtain

$$a\|u_0\|^2 = \lambda \int_{\Omega} |u_0|^q dx + \delta \int_{\Omega} |u_0|^4 dx$$

and

$$\bar{I}_0(u_0) = \frac{q-2}{2q}a\|u_0\|^2 + \frac{4-q}{4q}\delta \int_{\Omega} |u_0|^4 dx. \quad (5.2)$$

Moreover, u_0 is a mountain pass solution and \bar{I}_0 possesses the following properties:

- (M1) there exist $c, r > 0$ such that if $\|u\| = r$ then $\bar{I}_0(u) \geq c$ and there exists $v_0 \in H_0^1(\Omega)$ such that $\|v_0\| > r$ and $\bar{I}_0(v_0) < 0$;
- (M2) there exists a critical point $u_0 \in H_0^1(\Omega)$ of \bar{I}_0 such that

$$\bar{I}_0(u_0) = c_0 := \min_{\gamma \in \Gamma} \max_{t \in [0,1]} \bar{I}_0(\gamma(t)),$$

where $\Gamma := \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0, \gamma(1) = v_0\}$;

- (M3) $0 < c_0 := \inf\{\bar{I}_0(u) : \bar{I}'_0(u) = 0, u \in H_0^1(\Omega) \setminus \{0\}\} < (a\mathcal{S})^2/(4\delta)$;
- (M4) the set $\bar{\mathcal{S}} := \{u \in H_0^1(\Omega) : \bar{I}'_0(u) = 0, \bar{I}_0(u) = c_0\}$ is compact in $H_0^1(\Omega)$;
- (M5) there exists a path $\gamma_0 \in \Gamma$ passing through u_0 at $t = t_0$ and satisfying

$$\bar{I}_0(u_0) > \bar{I}_0(\gamma_0(t)) \quad \text{for all } t \neq t_0.$$

In fact, we can take $v_0 = Tu_0$ with $T > \sqrt{q/2}$ in (M1), and set $\gamma_0(t) = tv_0$ with $t_0 = 1/T$ in (M5). It is easy to verify that (M2) and (M3) hold. To check (M4), let $\{u_n\} \subset \bar{S}$ be a sequence, we deduce from (5.2) and the definition of \bar{S} that $\{u_n\}$ is a bounded (PS) sequence of \bar{I}_0 at the level c_0 . Noting that $c_0 < (a\mathcal{S})^2/(4\delta)$, by using the arguments of [6, 28], we see that \bar{I}_0 satisfies the Palais-Smale condition, and then there exists $\bar{u}_0 \in H_0^1(\Omega) \setminus \{0\}$ such that $u_n \rightarrow \bar{u}_0$ in $H_0^1(\Omega)$ as $n \rightarrow \infty$, so that $\bar{I}_0(\bar{u}_0) = c_0$ and $\bar{u}_0 \in \bar{S}$.

A nature way to construct a $(PS)_c$ sequence of \bar{I}_b with $c < (a\mathcal{S})^2/(4(\delta - b\mathcal{S}^2))$ can be processed as follows: let $v_0 \in H_0^1(\Omega)$ be as in (M1), since $\bar{I}_b(v_0) < 0$ for $b > 0$ small enough, it is easy to show that \bar{I}_b has the mountain pass geometry and hence there exists a sequence $\{u_n\} \subset H_0^1(\Omega)$ satisfying $\bar{I}_b(u_n) \rightarrow \bar{c}_b$ and $\bar{I}'_b(u_n) \rightarrow 0$ in $H^{-1}(\Omega)$, where

$$\bar{c}_b = \min_{\gamma \in \Gamma} \max_{t \in [0,1]} \bar{I}_b(\gamma(t))$$

and Γ is defined in (M2). Let $b > 0$ be so small that we have the following estimate of \bar{c}_b :

$$\begin{aligned} \bar{c}_b &\leq \max_{t \in [0,T]} \bar{I}_b(tu_0) \leq \max_{t \in [0,T]} \bar{I}_0(tu_0) + \frac{bT^4}{4} \|u_0\|^4 \\ &= c_0 + \frac{bT^4}{4} \|u_0\|^4 \\ &< \frac{(a\mathcal{S})^2}{4\delta} < \frac{(a\mathcal{S})^2}{4(\delta - b\mathcal{S}^2)}. \end{aligned}$$

However, we are not sure whether the $(PS)_{\bar{c}_b}$ sequence of the functional \bar{I}_b has a convergent subsequence in $H_0^1(\Omega)$ or not because the exponent of the nonlocal term $\|u\|^4$ is equal to the critical Sobolev exponent when $N = 4$, which leads to the difficulty of proving the boundedness of $\{u_n\}$.

We borrow ideas of [7, 12, 13, 20] and define a modified mountain pass level of \bar{I}_b by

$$c_b := \min_{\gamma \in \Gamma_M} \max_{t \in [0,1]} \bar{I}_b(\gamma(t))$$

where

$$\Gamma_M := \{\gamma \in \Gamma : \sup_{t \in [0,1]} \|\gamma(t)\| \leq M\} \text{ with } M = 2T\|u_0\|^2. \quad (5.3)$$

It follows from the choice of M that $\gamma_0 \in \Gamma_M$ and $c_0 = \min_{\gamma \in \Gamma_M} \max_{t \in [0,1]} \bar{I}_0(\gamma(t))$, where c_0 and γ_0 are respectively given in (M3) and (M5). Moreover, we have the following result:

Lemma 5.1 *The mountain pass level c_b is continuous at 0, i.e., $\lim_{b \rightarrow 0} c_b = c_0$.*

Proof. Clearly, $c_b \geq c_0$ holds. It follows from (M5) that

$$\begin{aligned} c_b &\leq \max_{1 \leq t \leq 1} \bar{I}_b(tv_0) \leq \max_{1 \leq t \leq T} \bar{I}_b(tu_0) + \frac{bT^4}{4} \|u_0\|^4 \\ &= c_0 + \frac{bT^4}{4} \|u_0\|^4 \\ &= c_0 + o(1) \end{aligned}$$

as $b \rightarrow 0$. ■

Given $d > 0$, a set $A \subset H_0^1(\Omega)$ and a point $u \in H_0^1(\Omega)$, we denote

$$B_d(u) := \{v \in H_0^1(\Omega) : \|v - u\| \leq d\}, \quad A^d := \bigcup_{u \in A} B_d(u).$$

Lemma 5.2 *Let \overline{S} be given in (M4). For fixed $d > 0$, if a sequence $\{u_j\} \subset \overline{S}^d$ then $\{u_j\}$ converges weakly to some $u \in \overline{S}^{2d}$ as $j \rightarrow \infty$, up to a subsequence.*

Proof. By the definition of \overline{S}^d , there exists $\{v_j\} \subset \overline{S}$ such that $\|u_j - v_j\| \leq d$ for all $j \in \mathbb{N}$. It follows from the compactness that there exists $v \in \overline{S}$ such that $\{v_j\}$ converges strongly to v in $H_0^1(\Omega)$ as $j \rightarrow \infty$, up to a subsequence, so that $\|u_j - v\| \leq 2d$ for large j , that is, $\{u_j\} \subset B_{2d}(v)$ for j large enough. Since $B_{2d}(v)$ is weakly closed in $H_0^1(\Omega)$ and $\{u_j\}$ is clearly bounded in $H_0^1(\Omega)$, we get that there exists $u \in \overline{S}^{2d}$ such that $u_j \rightharpoonup u$ as $j \rightarrow \infty$, up to a subsequence. \blacksquare

Remark 5.1 By the definition of \overline{S} in (M4), we see that there exists a positive constant C such that $\|u\| \geq C$ for all $u \in \overline{S}$ since, otherwise, we must have a sequence $\{u_n\} \subset \overline{S}$ such that $u_n \rightarrow 0$ in $H_0^1(\Omega)$ as $n \rightarrow \infty$, which yields $c_0 = 0$, a contradiction with the definition of c_0 . Therefore, we may choose a $d > 0$ so small that $u \neq 0$ for all $u \in \overline{S}^{2d}$.

Lemma 5.3 *Assume that $\{b_j\}$ is a sequence of positive numbers with $b_j \rightarrow 0$ as $j \rightarrow \infty$. Let $d > 0$ be small enough and let $\{u_j\} \subset \overline{S}^d$ be such that*

$$\lim_{j \rightarrow \infty} \overline{I}_{b_j}(u_j) \leq c_0, \quad \lim_{j \rightarrow \infty} \overline{I}'_{b_j}(u_j) = 0. \quad (5.4)$$

Then $\{u_j\}$ strongly converges to some $u \in \overline{S}$, up to a subsequence.

Proof. By Lemma 5.2 and Remark 5.1, up to a subsequence, we get that there exists $u \in \overline{S}^{2d}$ such that $u_j \rightharpoonup u$ in $H_0^1(\Omega)$ as $j \rightarrow \infty$ and $u \neq 0$. It can be deduce from (5.4) and the boundedness of $\{u_j\}$ that for all $v \in H_0^1(\Omega)$,

$$\begin{aligned} \overline{I}'_0(u)v &= a \int_{\Omega} \nabla u \nabla v dx - \lambda \int_{\Omega} |u|^{q-2} u v dx - \delta \int_{\Omega} |u|^2 u v dx \\ &= \lim_{j \rightarrow \infty} a \int_{\Omega} \nabla u_j \nabla v dx - \lambda \int_{\Omega} |u_j|^{q-2} u_j v dx - \delta \int_{\Omega} |u_j|^2 u_j v dx \\ &= \lim_{j \rightarrow \infty} \overline{I}'_{b_j}(u_j)v - b_j \|u_j\|^2 \int_{\Omega} \nabla u_j \nabla v dx = 0, \end{aligned}$$

hence $\overline{I}'_0(u) = 0$. Furthermore, since $\{u_j\} \subset \overline{S}^d$, we obtain that

$$\overline{I}'_0(u_j) = \overline{I}'_{b_j}(u_j) - b_j \|u_j\|^2 u_j = o(1).$$

On the other hand,

$$c_0 \geq \lim_{j \rightarrow \infty} \overline{I}_{b_j}(u_j) = \lim_{j \rightarrow \infty} \overline{I}_0(u_j) + \lim_{j \rightarrow \infty} b_j \|u_j\|^4 = \lim_{j \rightarrow \infty} \overline{I}_0(u_j) := m, \quad (5.5)$$

then $\{u_j\}$ is a $(PS)_m$ sequence of \bar{I}_0 . Therefore, up to a subsequence,

$$\begin{aligned}
\bar{I}_0(u) &= \frac{a}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{\lambda}{q} \int_{\Omega} |u|^q dx - \frac{\delta}{4} \int_{\Omega} |u|^4 dx \\
&= \frac{q-2}{2q} a \int_{\Omega} |\nabla u|^2 dx + \frac{4-q}{4q} \delta \int_{\Omega} |u|^4 dx \\
&\leq \lim_{j \rightarrow \infty} \frac{q-2}{2q} a \int_{\Omega} |\nabla u_j|^2 dx + \frac{4-q}{4q} \delta \int_{\Omega} |u_j|^4 dx \\
&= \lim_{j \rightarrow \infty} (\bar{I}_0(u_j) - \frac{1}{q} \bar{I}'_0(u_j) u_j) \\
&= m.
\end{aligned}$$

It follows from (M3) and (5.5) that $m = \bar{I}_0(u) = c_0$, which implies that $u \in \bar{S}$. A standard argument (cf. [6, 28]) shows that $u_j \rightarrow u$ in $H_0^1(\Omega)$ as $j \rightarrow \infty$. \blacksquare

Now we set $D_b := \max\{\bar{I}_b(\gamma_0(t)) : 0 \leq t \leq 1\}$. Clearly, $c_b \leq D_b$ and moreover, it holds $\lim_{b \rightarrow 0} D_b \leq c_0$, so that, by Lemma 5.1, we have

$$\lim_{b \rightarrow 0} c_b = \lim_{b \rightarrow 0} D_b = c_0. \quad (5.6)$$

The following Lemmas 5.4-5.6 are quite analogous to Propositions 3-5 in [20]. However, some arguments in [20] are based on the assumptions that the perturbation operator is compact, which is not our case here, therefore we prefer to give the modified proofs for sake of completeness.

Lemma 5.4 *Let $d_1 > d_2 > 0$ be two small constants, then there exist positive numbers α and b_0 depending on d_1 and d_2 such that for each $b \in (0, b_0)$, the following is true:*

$$\|\bar{I}'_b(u)\| \geq \alpha \quad \text{for all } u \in \bar{I}_b^{D_b} \cap (\bar{S}^{d_1} \setminus \bar{S}^{d_2}),$$

where $\bar{I}_b^{D_b} = \{u \in H_0^1(\Omega) : \bar{I}_b(u) \leq D_b\}$ as usual.

Proof. Argue by contradiction, we may suppose that there exist a sequence $\{b_j\}$ of positive numbers with $\lim_{j \rightarrow \infty} b_j = 0$ and a sequence of functions $\{u_j\} \subset \bar{I}_{b_j}^{D_{b_j}} \cap (\bar{S}^{d_1} \setminus \bar{S}^{d_2})$ such that $\lim_{j \rightarrow \infty} \bar{I}'_{b_j}(u_j) = 0$. Then by (5.6), we obtain that $\lim_{j \rightarrow \infty} \bar{I}_{b_j}(u_j) \leq c_0$. It follow from Lemma 5.3 that there exists $u \in \bar{S}$ such that $u_j \rightarrow u$ in $H_0^1(\Omega)$. As a consequence, $\text{dist}(u_j, \bar{S}) \rightarrow 0$ as $j \rightarrow \infty$, this contradict $u_j \notin \bar{S}^{d_2}$. \blacksquare

Lemma 5.5 *Let $d > 0$ be fixed. Then there exists $\delta > 0$ such that if $b > 0$ small enough,*

$$t \in [0, 1] \text{ and } \bar{I}_b(\gamma_0(t)) \geq c_b - \delta \Rightarrow \gamma_0(t) \in \bar{S}^d.$$

Proof. Since the proof is quite similar to that of [20]. We omit the details. \blacksquare

Lemma 5.6 *Let $d > 0$ be a small constant. Then there exist $b > 0$ sufficiently small, depending on d , and a sequence $\{u_j\} \subset \bar{S}^d \cap \bar{I}_b^{D_b}$ such that $\bar{I}'_b(u_j) \rightarrow 0$ as $j \rightarrow \infty$.*

Proof. Arguing by contradiction, we may assume that there exist two sequences $\{b_j\}$ and $\{c_j\}$ of positive numbers with $b_j \rightarrow 0$ such that $\|\bar{T}'_{b_j}(u)\| \geq c_j > 0$ for all $u \in \bar{S}^d \cap \bar{T}_{b_j}^{D_{b_j}}$. By Lemmas 5.3 and 5.4, there exists some $\alpha > 0$, independent of j , such that

$$\|\bar{T}'_{b_j}(u)\| \geq \alpha \text{ for all } u \in \bar{T}_{b_j}^{D_{b_j}} \cap (\bar{S}^d \setminus \bar{S}^{\frac{d}{2}}),$$

which gives that $\{c_j\}$ has a positive lower bound. Moreover, there exists $k > 0$, independent of j , such that $\|\bar{T}'_{b_j}(u)\| \leq k$ for all $u \in \bar{S}^d$ since \bar{S}^d is bounded in $H_0^1(\Omega)$. By using Lemma 5.5 and (5.6), we can choose $\delta > 0$ so small and j so large that

$$t \in [0, 1], \bar{T}_{b_j}(\gamma_0(t)) \geq c_b - \frac{\delta}{4} \text{ implies } \gamma_0(t) \in \bar{S}^{\frac{d}{2}} \quad (5.7)$$

and

$$D_{b_j} - c_{b_j} < \frac{\delta}{4} \text{ and } D_{b_j} - c_{b_j} < \frac{\alpha^2 d}{4k} - \frac{\delta}{4}. \quad (5.8)$$

From now on, we fix a j so large that (5.7) and (5.8) are true, and we denote b_j just by b . Now, consider a pseudo-gradient vector field V_b of \bar{T}_b and a neighborhood \mathcal{N}_b of $\bar{S}^d \cap \bar{T}_b^{D_b}$ satisfying $\mathcal{N}_b \subset B_M(0)$, where M is given by (5.3). We observe that $D_b < M$ for b small enough.

Let us respectively define two functions $\eta_b \in C^{1,1}(H_0^1(\Omega), [0, 1])$ and $\xi_b \in C^{1,1}(\mathbb{R}, [0, 1])$ by

$$\eta_b = \begin{cases} 1, & \text{on } \bar{S}^d \cap \bar{T}_b^{D_b}, \\ 0, & \text{on } H_0^1(\Omega) \setminus \mathcal{N}_b \end{cases}$$

and

$$\xi_b(t) = \begin{cases} 1, & \text{if } |t - c_b| \leq \frac{\delta}{2}, \\ 0, & \text{if } |t - c_b| \geq \delta. \end{cases}$$

Then it is clear that the following Cauchy initial value problem

$$\begin{cases} \frac{d}{dt} \psi_b(u, t) = -\eta_b(\psi_b(u, t)) \xi_b(\bar{T}_b(\psi_b(u, t))) V_b(\psi_b(u, t)), \\ \psi_b(u, 0) = u \end{cases}$$

has a global solution $\psi_b : H_0^1(\Omega) \times \mathbb{R} \rightarrow H_0^1(\Omega)$ since η_b and $\xi_b \circ \bar{T}_b \in C^{1,1}(H_0^1(\Omega), [0, 1])$ and V_b is locally Lipschitz continuous.

Now we claim that for all $t \in [0, 1]$, $\bar{T}_b(\psi_b(\gamma_0(t), t_b)) \leq c_b - \delta/4$, where $t_b := \delta/(2c^2)$. Indeed, if for each $t \in [0, 1]$, there exists $t_0 \leq t_b$ such that $\bar{T}_b(\psi_b(\gamma_0(t), t_0)) \leq c_b - \delta/4$, then we get the claim since it follows from

$$\frac{d}{d\tau} \bar{T}_b(\psi_b(\gamma_0(t), \tau)) \leq -\eta_b(\psi_b(\gamma_0(t), \tau)) \xi_b(\bar{T}_b(\psi_b(\gamma_0(t), \tau))) \|\bar{T}'_b(\psi_b(\gamma_0(t), \tau))\|^2 \quad (5.9)$$

that

$$\bar{T}_b(\psi_b(\gamma_0(t), t_b)) \leq \bar{T}_b(\psi_b(\gamma_0(t), t_0)) \leq c_b - \frac{\delta}{4}.$$

Otherwise, there exists some $t \in [0, 1]$ such that

$$\bar{T}_b(\psi_b(\gamma_0(t), \tau)) > c_b - \frac{\delta}{4} \text{ for all } \tau \in [0, t_b]. \quad (5.10)$$

Thus by (5.7), we have

$$\gamma_0(t) = \psi_b(\gamma_0(t), 0) \in \overline{S}^{\frac{d}{2}} \text{ and } \xi_b(\overline{I}_b(\psi_b(\gamma_0(t), \tau))) = 1 \text{ for all } \tau \in [0, t_b].$$

If $\psi_b(\gamma_0(t), \tau) \in \overline{S}^d$ ($\forall \tau \in [0, t_b]$) then it follows from (5.8) and $\overline{I}_b(\psi_b(\gamma_0(t), \tau)) \leq \overline{I}_b(\gamma_0(t)) \leq D_b$ that $\eta_b(\psi_b(\gamma_0(t), \tau)) = 1$ ($\forall \tau \in [0, t_b]$), which, together with (5.9), implies that

$$\frac{d}{dt} \overline{I}_b(\psi_b(\gamma_0(t), \tau)) \leq -c^2,$$

where $c := c_j$ for the above fixed j . Thus we obtain

$$\overline{I}_b(\psi_b(\gamma_0(t), t_b)) = \int_0^{t_b} \frac{d}{dt} \overline{I}_b(\psi_b(\gamma_0(t), \tau)) d\tau + \overline{I}_b(\gamma_0(t)) \leq D_b - \int_0^{t_b} c^2 d\tau = D_b - t_b c^2 < c_b - \frac{\delta}{4},$$

a contradiction with (5.10). Therefore we must have $\psi_b(\gamma_0(t), \tau_0) \notin \overline{S}^d$ for some $\tau_0 \in [0, t_b]$, so that there exist $\tau_1, \tau_2 : 0 \leq \tau_1 < \tau_2 \leq \tau_0$ satisfying $\psi_b(\gamma_0(t), \tau_1) \in \partial \overline{S}^{\frac{d}{2}}$, $\psi_b(\gamma_0(t), \tau_2) \in \partial \overline{S}^d$ and $\psi_b(\gamma_0(t), \tau) \in \overline{S}^d \setminus \overline{S}^{\frac{d}{2}}$ for all $\tau \in (\tau_1, \tau_2)$. It is not hard to show that $\tau_2 - \tau_1 \geq d/(4k)$ since, otherwise, we shall get, by the definition of pseudo-gradient vector field, that

$$\begin{aligned} \frac{d}{2} &= \|\psi_b(\gamma_0(t), \tau_2) - \psi_b(\gamma_0(t), \tau_1)\| = \left\| \int_{\tau_1}^{\tau_2} \frac{d}{d\tau} \psi_b(\gamma_0(t), \tau) d\tau \right\| \\ &\leq \int_{\tau_1}^{\tau_2} \left\| \frac{d}{d\tau} \psi_b(\gamma_0(t), \tau) \right\| d\tau \leq \int_{\tau_1}^{\tau_2} \|I'_b(\psi_b(\gamma_0(t), \tau))\| d\tau \leq \int_{\tau_1}^{\tau_2} 2kd\tau < \frac{d}{2}, \end{aligned}$$

this is impossible. Thus, from (5.8) and (5.9), we obtain

$$\begin{aligned} \overline{I}_b(\psi_b(\gamma_0(t), t_b)) &\leq \overline{I}_b(\psi_b(\gamma_0(t), \tau_2)) = \int_0^{\tau_2} \frac{d}{d\tau} \overline{I}_b(\psi_b(\gamma_0(t), \tau)) d\tau + \overline{I}_b(\gamma_0(t)) \\ &\leq \int_{\tau_1}^{\tau_2} \frac{d}{d\tau} \overline{I}_b(\psi_b(\gamma_0(t), \tau)) d\tau + \overline{I}_b(\gamma_0(t)) \leq D_b - \alpha^2(\tau_2 - \tau_1) \\ &< D_b - \frac{d\alpha^2}{4k} < c_b - \frac{\delta}{4}, \end{aligned}$$

a contradiction with (5.10) again. Hence we have prove the claim.

Finally, we set $\tilde{\gamma}_0(t) := \psi_b(\gamma_0(t), t_b)$. Then $\tilde{\gamma}_0(t) \in \Gamma_M$ and $\overline{I}_b(\tilde{\gamma}_0(t)) < c_b$ for all $t \in [0, 1]$, this contradicts the definition of c_b . Thus we complete the proof of this lemma. \blacksquare

Proof of Theorem 1.4:

Let us fix $d > 0$ small enough. By Lemma 5.6, (5.3) and (M3), we deduce that there exist $b > 0$ sufficiently small and a (PS) sequence $\{u_n\}$ of \overline{I}_b with $\{u_n\} \subset \overline{S}^{\frac{d}{2}} \cap \overline{I}_b^{D_b}$ and $D_b < (a\mathcal{S})^2/(4\delta) < (a\mathcal{S})^2/(4(\delta - b\mathcal{S}^2))$. It follows from the (5.2), (M3) and the definition of \overline{S}^d that $\{u_n\}$ is bounded in $H_0^1(\Omega)$, in particular, $\|u_n\|^2 \leq 1 + (aq\mathcal{S}^2)/(2\delta(q-2))$ for all $n \in \mathbb{N}$. Therefore, there exists $c > 0$ with $c < (a\mathcal{S})^2/(4\delta) < (a\mathcal{S})^2/(4(\delta - b\mathcal{S}^2))$ such that $\overline{I}_b(u_n) \rightarrow c$, up to a subsequence. Next we will show that there exists $\overline{u}_0 \in \overline{S}^d$ such that $u_n \rightarrow \overline{u}_0$ in $H_0^1(\Omega)$ as $n \rightarrow \infty$, up to a subsequence, provided one of the three conditions (C1)-(C3) holds. Arguing by contradiction, if $\{u_n\}$ has no convergent subsequence in $H_0^1(\Omega)$ then by Proposition 2.1, there exist a weak limit $\overline{u}_0 \in H_0^1(\Omega)$ of $\{u_n\}$, a number $k \in \mathbb{N}$ and further, for every $i \in \{1, 2, \dots, k\}$, a sequence of values $\{R_n^i\} \subset \mathbb{R}^+$, points $\{x_n^i\} \subset \overline{\Omega}$ and a function $v_i \in \mathcal{D}^{1,2}(\mathbb{R}^4)$ satisfying (2.6), (2.7) and

$$\overline{I}_b(u_n) = \tilde{I}_b(\overline{u}_0) + \sum_{i=1}^k \tilde{I}_b^\infty(v_i) + o(1)$$

up to a subsequence, where $o(1) \rightarrow 0$ as $n \rightarrow \infty$, \tilde{I}_b and \tilde{I}_b^∞ are respectively given by (2.8) and (2.9). On one hand, by (2.6) and (2.8), we have

$$\begin{aligned}\tilde{I}_b(\bar{u}_0) &= \tilde{I}_b(\bar{u}_0) - \frac{1}{4} \left((a + b(\|\bar{u}_0\|^2 + \sum_{i=1}^k \|v_i\|_{\mathcal{D}^{1,2}}^2)) \|\bar{u}_0\|^2 - \lambda \int_{\Omega} |\bar{u}_0|^q dx - \delta \int_{\Omega} |\bar{u}_0|^4 dx \right) \\ &= \frac{a}{4} \|\bar{u}_0\|^2 - \frac{(4-q)\lambda}{4q} \int_{\Omega} |\bar{u}_0|^q dx \\ &\geq \frac{a}{4} \|\bar{u}_0\|^2 - \frac{(4-q)\lambda|\Omega|^{\frac{4-q}{4}}}{4q\mathcal{S}^{q/2}} \|\bar{u}_0\|^q.\end{aligned}\tag{5.11}$$

On the other hand, it can be deduced from (2.9) that for $j \in \{1, 2, \dots, k\}$,

$$\begin{aligned}0 &= (a + b(\|\bar{u}_0\|^2 + \sum_{i=1}^k \|v_i\|_{\mathcal{D}}^2)) \|v_j\|_{\mathcal{D}}^2 - \delta \int_{\mathbb{R}^4} |v_j|^4 dx \\ &\geq (a + b\|\bar{u}_0\|^2) \|v_j\|_{\mathcal{D}}^2 - \frac{\delta - b\mathcal{S}^2}{\mathcal{S}^2} \|v_j\|_{\mathcal{D}}^4,\end{aligned}$$

which implies that

$$\|v_j\|_{\mathcal{D}^{1,2}}^2 \geq \frac{\mathcal{S}^2(a + b\|\bar{u}_0\|^2)}{\delta - b\mathcal{S}^2}.$$

Therefore, by (2.7), we obtain that

$$\begin{aligned}\tilde{I}_b^\infty(v_j) &= \tilde{I}_b^\infty(v_j) - \frac{1}{4} (a + b(\|\bar{u}_0\|^2 + \sum_{i=1}^k \|v_i\|_{\mathcal{D}}^2)) \|v_j\|_{\mathcal{D}}^2 - \delta \int_{\mathbb{R}^4} |v_j|^4 dx \\ &= \frac{a}{4} \|v_j\|_{\mathcal{D}}^2 \geq \frac{a\mathcal{S}^2(a + b\|\bar{u}_0\|^2)}{4(\delta - b\mathcal{S}^2)}.\end{aligned}$$

this, together with (5.11), yields that

$$\begin{aligned}c &= \lim_{n \rightarrow \infty} \bar{I}_b(u_n) \geq \tilde{I}_b(\bar{u}_0) + \tilde{I}_b^\infty(v_i) \\ &= \frac{a}{4} \|\bar{u}_0\|^2 - \frac{(4-q)\lambda}{4q\mathcal{S}^{q/2}} \|\bar{u}_0\|^q + \frac{(a\mathcal{S})^2}{4(\delta - b\mathcal{S}^2)} + \frac{ab\mathcal{S}^2}{4(\delta - b\mathcal{S}^2)} \|\bar{u}_0\|^2 \\ &= \frac{(a\mathcal{S})^2}{4(\delta - b\mathcal{S}^2)} + \left(\frac{a\delta}{4(\delta - b\mathcal{S}^2)} - \frac{(4-q)\lambda}{4q\mathcal{S}^{q/2}} \|\bar{u}_0\|^{q-2} \right) \|\bar{u}_0\|^2 \\ &\geq \frac{(a\mathcal{S})^2}{4(\delta - b\mathcal{S}^2)} + \left(\frac{a\delta}{4(\delta - b\mathcal{S}^2)} - \frac{(4-q)\lambda|\Omega|^{\frac{4-q}{4}}}{4q\mathcal{S}^{q/2}} \left(1 + \frac{aq\mathcal{S}^2}{2\delta(q-2)} \right)^{\frac{q-2}{2}} \right) \|\bar{u}_0\|^2,\end{aligned}$$

here we have used the inequality $\|\bar{u}_0\|^2 \leq 1 + (aq\mathcal{S}^2)/(2\delta(q-2))$ since $\|u_n\|^2 \leq 1 + (aq\mathcal{S}^2)/(2\delta(q-2))$ and \bar{u}_0 is the weak limit of u_n in $H_0^1(\Omega)$.

Now, recalling that $2 < q < 4$, it follows from one of the assumptions (C1)-(C3) that

$$c \geq \frac{(a\mathcal{S})^2}{4(\delta - b\mathcal{S}^2)},$$

this is a contradiction since $c < (a\mathcal{S})^2/(4(\delta - b\mathcal{S}^2))$. Therefore, up to a subsequence, $\{u_n\}$ converges to some $\bar{u}_0 \in H_0^1(\Omega)$. Noting that $\bar{u}_0 \neq 0$ (cf. Remark 5.1), we know that \bar{u}_0 is a desired solution of Eq. (1.1). \blacksquare

Proof of Theorem 1.5:

(i) We suppose on the contrary that Eq. (1.1) has a solution $u \in H_0^1(\Omega) \setminus \{0\}$ under assumptions $\delta < b\mathcal{S}^2$ and (1.7), that is, $\bar{I}'_b(u)v = 0$ for all $v \in H_0^1(\Omega)$, so that

$$\bar{I}'_b(u)u = a\|u\|^2 + b\|u\|^4 - \lambda\|u\|_q^q - \delta\|u\|_4^4 = 0. \quad (5.12)$$

For $t > 0$, we set $f(t) = \bar{I}_b(tu)$, then

$$f'(t) = at\|u\|^2 + t^3(b\|u\|^4 - \delta\|u\|_4^4) - \lambda t^{q-1}\|u\|_q^q := th(t),$$

where $h(t) = a\|u\|^2 + t^2(b\|u\|^4 - \delta\|u\|_4^4) - \lambda t^{q-2}\|u\|_q^q$. It follows from (5.12) that $f'(1) = 0$. On the other hand, it is easy to see that

$$\bar{t} = \left(\frac{\lambda(q-2)\|u\|_q^q}{2(b\|u\|^4 - \delta\|u\|_4^4)} \right)^{\frac{1}{4-q}}$$

is the unique root of equation $h'(t) = 0$ on $(0, +\infty)$, which, together with the Hölder inequality, the inequality (1.7) and the fact of $q \in (2, 4)$, implies

$$\begin{aligned} \min_{t>0} h(t) &= h(\bar{t}) = a\|u\|^2 + \bar{t}^2(b\|u\|^4 - \delta\|u\|_4^4) - \lambda\bar{t}^{q-2}\|u\|_q^q \\ &= a\|u\|^2 - \frac{(4-q)}{2}\lambda\|u\|_q^q \left(\frac{\lambda(q-2)\|u\|_q^q}{2(b\|u\|^4 - \delta\|u\|_4^4)} \right)^{\frac{q-2}{4-q}} \\ &\geq a\|u\|^2 - \frac{(4-q)}{2} \left(\frac{(q-2)\mathcal{S}^2}{2(b\mathcal{S}^2 - \delta)} \right)^{\frac{q-2}{4-q}} \frac{\lambda^{\frac{2q}{4-q}}\|u\|_q^{\frac{2q}{4-q}}}{\|u\|^{\frac{4(q-2)}{4-q}}} \\ &\geq \|u\|^2 \left(a - (4-q) \left(\frac{\lambda}{2} \right)^{\frac{2}{4-q}} \left(\frac{|\Omega|^{\frac{4-q}{4}}}{\mathcal{S}^{\frac{q}{2}}} \right)^{\frac{2}{4-q}} \left(\frac{(q-2)\mathcal{S}^2}{b\mathcal{S}^2 - \delta} \right)^{\frac{q-2}{4-q}} \right) \geq 0, \end{aligned}$$

this gives that $f'(t) \geq 0$ for all $t \geq 0$. Clearly $f'(t) > 0$ for $t > 0$ small enough, thus we obtain that $0 = f(0) < f(1) = 0$, a contradiction.

(ii) We also use

$$\bar{I}_0(u) = \frac{a}{2}\|u\|^2 - \frac{\lambda}{q}\|u\|_q^q, \quad u \in H_0^1(\Omega) \quad (5.13)$$

to denote the limited functional of \bar{I}_b as $b \rightarrow 0^+$ in the following proof. Clearly, $\bar{I}_0 \in C^2(H_0^1(\Omega), \mathbb{R})$ and critical points of \bar{I}_0 are weak solutions of the following equation

$$\begin{cases} -a\Delta u = \lambda|u|^{q-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Noting that if $0 < \delta < b\mathcal{S}^2$ then for all $u \in H_0^1(\Omega) \setminus \{0\}$ we have

$$\begin{aligned} \bar{I}_b(u) &= \frac{a}{2}\|u\|^2 + \frac{1}{4}(b\|u\|^4 - \delta\|u\|_4^4) - \frac{\lambda}{q}\|u\|_q^q \\ &\geq \frac{a}{2}\|u\|^2 + \frac{1}{4}(b - \frac{\delta}{\mathcal{S}^2})\|u\|^4 - \frac{\lambda}{q}\|u\|_q^q \rightarrow +\infty \end{aligned}$$

as $\|u\| \rightarrow +\infty$, therefore \bar{I}_b is coercive. On the other hand, for given $a > 0$ and $\lambda > 0$, we can find $v_0 \in H_0^1(\Omega) \setminus \{0\}$ such that $\bar{I}_0(v_0) < 0$, then there exists $\bar{b}_1 > 0$ dependent of a and λ such that for each $b \in (0, \bar{b}_1)$, $\bar{I}_b(v_0) < 0$, therefore, $\bar{d} := \inf\{\bar{I}_b(u) : u \in H_0^1(\Omega)\} < 0$. By Ekeland variational principle, there exists a sequence $\{u_n\}$ satisfying $\bar{I}_b(u_n) \rightarrow \bar{d} < 0$ and $\bar{I}'_b(u_n) \rightarrow 0$ strongly in

$H^{-1}(\Omega)$. It follows from Proposition 2.1 and (2.5) that there exists $\bar{u}_1 \in H_0^1(\Omega) \setminus \{0\}$ such that $u_n \rightarrow \bar{u}_1$ in $H_0^1(\Omega)$, thus $\bar{I}'_b(\bar{u}_1) = 0$, so that \bar{u}_1 is a solution of (1.1). To obtain the second solution, we just remind that the functional \bar{I}_0 defined by (5.13) has the similar properties (M1) – (M5), then by following the proof of Theorem 1.4 and combining with Proposition 2.1 and (2.5), we can get that there exists $\bar{b}_2 > 0$ dependent of a and λ such that for each $b \in (0, \bar{b}_2)$, Eq. (1.1) has a mountain pass solution $u_2 (\neq u_1)$ in both cases of $\delta < b\mathcal{S}^2$ and $\delta = b\mathcal{S}^2$, here we omit the details of proof. ■

Remark 5.2 \bar{I}_b satisfies the $(PS)_c$ condition for all $c \in \mathbb{R}$ if the $(PS)_c$ sequence is bounded, this can be deduced easily from Proposition 2.1 and (2.5).

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